

# Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 07

## State Space analysis of Aircraft longitudinal dynamics

Hello everyone, in today's lecture, we will be discussing how we can apply the concepts of what we have done in the last lecture to the aircraft system. So, here we are going to consider the longitudinal motion of the aircraft and how we can study the state-transition matrix, eigenvalues, eigenvectors, and how we can analyze the system in open loop. So, based on this, we can go through some control design techniques later. So, this is a very important part before we move to the control design. So, let us write the longitudinal motion of the aircraft.

So, we could write the equation. Let us list the topics we will cover in the lecture. So, first, we are going to do motion analysis. of the longitudinal aircraft dynamics and define it in state-space form. state-space form And also study and analyze the system based on the state matrix, eigenvalues,

eigenvectors. So, these components are very important before we move to the control design part. We need to study the system and understand how the system is going to behave. So, here I will write the equation, the equation we obtained in our classical control course. So, here the parameters involved in the longitudinal motion are  $u$ ,  $w$ ,

$\dot{u}$  and  $\dot{\theta}$ . So, let me write the equation what we obtained it is very standard equation Systematics, this is the example for, this example actually, Lockheed Martin Aircraft, Lockheed F 104 Aircraft, so this data we can find from this side. so this is the this the values the aerodynamic parameters stability derivatives we have taken for this particular aircraft located 104 aircraft so the system matrix obtained as

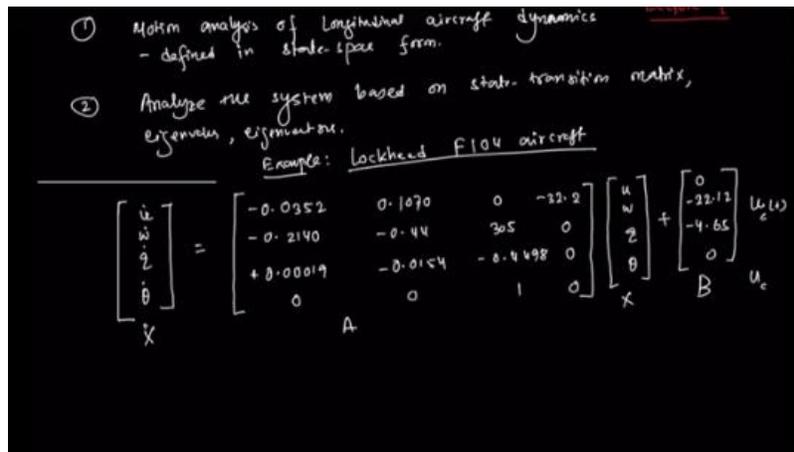
$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.0352 & 0.1070 & 0 & -32.2 \\ -0.2140 & -0.44 & 305 & 0 \\ 0.00019 & -0.0154 & -0.4498 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ -22.12 \\ -4.65 \\ 0 \end{bmatrix} U_c(t)$$

okay so this is basically I can write

$$\dot{X} = AX + BU_c$$

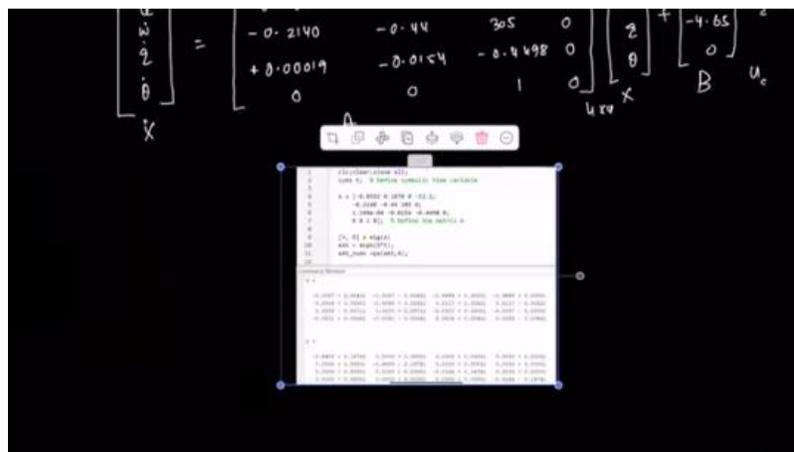
now this matrix you can easily derive if you are given the as stability derivatives parameters and aerodynamics parameters. So, if you substitute the long you know what the linearized model of the longitudinal motion of the aircraft you can get this state space form.

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So, now here this is the matrices 4 cross 4 matrix. So, doing everything analytically it is quite challenging. So, what you can use you can use the MATLAB code give the eigenvalues and eigenvectors so here this is the matlab code so here if you notice here this is our a matrix we have defined here and there is a command so if you use like if you write  $e = eig(A)$  so if you use this command you will get only the eigen values but instead of this if you use this command it will give the eigenvector basically the p matrix here we can write instead of p you can write this is p matrix and this is our diagonal matrix

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So, the diagonal matrix is nothing but D, and D is nothing but here we can write this as basically  $T^{-1}AT$ . This is basically the diagonal matrix, and here V is actually our transformation matrix, which is our—since the system dimension is 4 by 4—so we have four vectors:  $v_1, v_2, v_3$ , and  $v_4$ . So, this is basically the eigenvectors which form the V matrix here, okay? So, this is how we can find—this is, I can write—this is the T matrix, I can write. This is the T matrix, and this is the D matrix, which is nothing but T inverse, the diagonal matrix. So, if you notice, we have only the diagonal terms in our system; the off-diagonal terms are 0 here. Okay, so here we are going to do some analysis.

Since we are having—if you notice what we have done previously—we denoted this matrix. Also, we denoted it as A; also, we have noticed this. Also, given A dash, right? This is the same thing we have done in the previous lecture. So now, here we can write—let me rewrite this

$$A' = \begin{bmatrix} -0.4459 + 2.167i & 0 & 0 & 0 \\ 0 & -0.4459 - 2.167i & 0 & 0 \\ 0 & 0 & -0.0166 + 0.147i & 0 \\ 0 & 0 & 0 & -0.0166 - 0.147i \end{bmatrix}$$

here. If you notice here, it's zero—actually, the off-diagonal terms are zero here, okay? these are the complex conjugate roots. So, this is how the system is written in off-diagonal terms. So, what we have done in the previous lecture— so we can assume the system is written as in a different domain:

$$\dot{Z}(t) = A'Z(t)$$

So, this is our new system. So, here A' actually represents this matrix. Okay. So, now if you start with a system based on the diagonal elements here, the diagonal elements— what we have done is diagonal elements. So, if you notice, there are two complex conjugate roots: one is a complex conjugate pair, and this is another pair of complex conjugates.

So, if you notice here in this case, If you remember from our last lecture course, this talks about the damping part, and this talks about the natural frequency. This is also damping; this is also frequency. And in this complex conjugate root, this is the damping ratio, and this is the frequency. Similarly, this applies to this case as well.

So, what do you get here? So, here, In this complex conjugate root, since the frequency is higher, the period will be less. And in this complex root, the frequency is lower than this frequency, so automatically the period will be higher. Why?

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Handwritten mathematical derivation on a blackboard. At the top, it says  $A = T^{-1} A' T$ . Below that is a 4x4 matrix with diagonal elements:  $-0.4459 + 2.167i$ ,  $-0.4459 - 2.167i$ ,  $-0.0166 + 0.147i$ , and  $-0.0166 - 0.147i$ . The matrix is labeled  $A'$ . Below the matrix, the differential equation is written as  $\dot{Z}(t) = A' Z(t)$ .

Because the, how to find the period?

$$T = \frac{2\pi}{\omega}$$

So, omega if it is high, automatically period will be less. If omega goes to low value, the period will be less. So, in this case, easily you can comment, this is the, this part, this complex conjugate root yields the long period motion or phugoid motion.

And these complex conjugate roots yield the short period motion because in this case the period is less compared to this roots. So, we will go through detail about it. So, now let us talk the short period mode how it is coming short period case short period So, here we have the complex conjugate root. So, here

$$\lambda_{1,2} = -0.4459 \pm 2.1676i$$

So, since the roots are, there are two roots, so we can compare the system with our previously derived case so we can write any standard system some transfer function we can write

$$T(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

in this case also we have some second order equation

$$\begin{aligned}\lambda_{1,2} &= -\xi\omega_n \pm i\sqrt{1 - \xi^2} \\ &= -\xi\omega_n \pm i\omega_d \\ \omega_d &= \omega_n\sqrt{1 - \xi^2}\end{aligned}$$

So, here  $\xi$  is the damping ratio. Damping ratio and  $\omega_n$  is the natural frequency. And this whole term, this whole term, we can also write as damped natural frequency  $\omega_d$ . Not damped, sorry, not natural, it is the damped frequency we can define, damped frequency. Why damped frequency? Because the damping factor is also there in the function  $\omega_d$ , is the damped frequency. Now, if you compare this equation, if you compare this equation, and this equation.

So, because these are the roots we are getting from this characteristic equation, and these are the roots we are getting from our aircraft system. So, and these are the roots we are getting from the standard transfer function. This is the standard transfer function, right? So, now we can compare this result with this result and if you compare both the results, we can write

$$\begin{aligned}\xi\omega_n &= 0.4459 \\ \omega_n\sqrt{1 - \xi^2} &= 2.1676 \\ \xi &= 0.2014 \\ \omega_n &= 2.214\end{aligned}$$

Okay, this is, and this is the case we are getting for the short period case. So, this whole thing is for the short period case now.

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Short period  $\lambda_{1,2} = -0.0059 \pm 2.1676i$

$\Rightarrow \lambda_{1,2} = -\zeta \omega_n \pm i \omega_n \sqrt{1-\zeta^2}$

damping ratio  $\rightarrow$   $-\zeta \omega_n$

natural frequency  $\rightarrow$   $\omega_n$

damped frequency  $\rightarrow$   $\omega_n \sqrt{1-\zeta^2}$

$= -\zeta \omega_n \pm i \omega_d$  ( $\omega_d = \omega_n \sqrt{1-\zeta^2}$ )

the long period case or phugoid case similarly if you compare for the phugoid mode or long period case we can write one two so there are two complex conjugate roots which are basically

$$\lambda_{1,2} = -0.0166 \pm i0.1474$$

$$\xi = 0.11$$

$$\omega_n = 0.14$$

Now, if you find  $T = \frac{2\pi}{\omega}$ , this is the  $\omega$  if you fit here, we will get some numerical values. So, the value is what will you get something A and B. So, A is less than B. So, that is why. So, you can say it should be. So, period is less in short period case. Also, if you look the damping part, here the damping ratio is less, higher than the damping ratio what you get in the long pilot case. So in the system if it is high damping, we can diminish the sustained oscillation very quickly. This is the, this is very important way how we can analyze the motion dynamics, natural motion dynamics of the longitudinal motion of the aircraft. So I hope it is clear this part how we can define or how we can come up the conclusion whether the system is how it is stable whether taking more time to get stable because anyways the roots are all lies in the negative side I mean right hand side of the S plane if you see in the particular example we are having all the roots in the left hand side

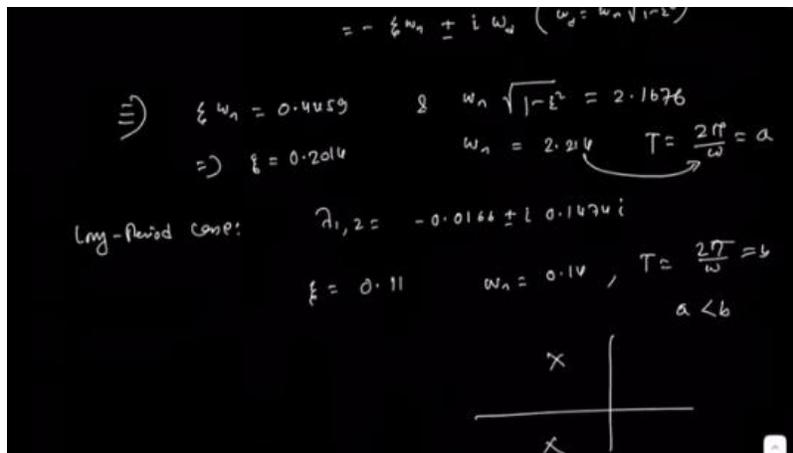
So, it means the system is already scheduled, but how it is going to behave over time. So, there are two important parts in this particular motion dynamics: one is the short period and long period, and how we can discuss them from the state transition matrix. So, this is our state transition matrix. The state transition matrix is also something we can find—sorry, state transition, actually this is the This is basically the system in the diagonal form, and how we can use the system which is in diagonalized form. We can use this matrix, this matrix, and we can do the analysis: the short-wave case and long-period case.

So, this is how we can analyze the system. Now, we will move to the next part, which is basically one of the most important concepts in control theory: the control matrix. Controllability is a very important part. Before you go to the design of control power or controller, we need to check whether the system is controllable or not. For example, we are having a space, for instance, and we have multiple states in the system:  $x_1, x_2$  of the projection, or we have the system in this form:

$$\dot{X} = AX + BU$$

And let us assume this  $x$  actually is the—there are a lot of states,  $x$  and states are there. Now, you are applying the control input to this system  $u$  of  $t$ . So, control  $t$  means if all the states are affected by this control input, then you can say the system is controllable.

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If any of the states is not affected by that control input, then you can say the system is not controllable. So, let us take a motivating example, then it will be clear. Let us take an example. So, we have system

$$\dot{X} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} X + \begin{bmatrix} 2 \\ 0 \end{bmatrix} U(t)$$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$Y = [3 \ 0]X$$

We will find the transfer function here. So, in this particular case, since the matrix is 2 into 2 dimensional, we may get as per the theory we have done, if the system is 2 plus 2 matrix, the characteristic equation will be of second order. The characteristic equation of the system should be second order. But let us check whether the characteristic equation is coming out to be second order or not. So, before that, we will move, we will go to the how to find the transfer function for this in the state-space model.

So, in the state-space model,  $DU$  part we are not considering here because we are not using any feed-through case, okay. If there is any feed-through, it means if there is any

control from the reference signal, then du part should be there, but it is not there, so du part we are adding to be 0. So, now we will take the Laplace transform of this equation, we will get

$$SX(S) = AX(S) + BU(S)$$

$$Y(S) = CX(S)$$

Okay. So, one thing please remember to find the transfer function of the system initial condition assume to be 0. If you assume non-zero initial condition, so we cannot apply the principle of superposition and system will be non-linear. So, for finding the transfer function the system should be, the system should have initial condition should have 0. This is the condition. So, now from this we can write.

$$(SI - A)X(S) = BU(S)$$

$$\frac{Y(S)}{U(S)} = C(SI - A)^{-1}B$$

So, this is the transfer function. If you use the state-space form, we can get this transfer function here. Okay. So, now we will use this method, how we can find the transfer function for this particular example. Okay and we can substitute the parameters values of the matrices. Actually, there is a typo. The system, the output matrix C matrix, It should be 2. So, Y is function of all the states, explanation.

$$\begin{aligned} \frac{Y(S)}{U(S)} &= [3 \quad 2] \left\{ \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \frac{6}{S+2} \end{aligned}$$

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Handwritten derivation on a blackboard:

$$\begin{aligned} \dot{X} &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} X + \begin{bmatrix} 2 \\ 0 \end{bmatrix} U(s) \\ Y &= \begin{bmatrix} 3 & 2 \end{bmatrix} X \end{aligned}$$

$$\frac{Y(s)}{U(s)} = C (sI - A)^{-1} B$$

$$= [3 \quad 2] \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= [3 \quad 2] \frac{1}{s+2}$$

Summary of steps on the right side of the board:

$$\begin{aligned} \dot{X} &= Ax + Bu \\ Y &= Cx \\ \downarrow \\ sX(s) &= Ax(s) + Bu(s) \\ Y(s) &= Cx(s) \\ (sI - A)X(s) &= Bu(s) \\ \Rightarrow (sI - A) \frac{Y(s)}{C} &= Bu(s) \\ \Rightarrow \frac{Y(s)}{U(s)} &= C (sI - A)^{-1} B \end{aligned}$$

So, what is the problem here? The problem is, as you say, the characteristic equation of the transfer function  $\frac{Y(s)}{U(s)}$ , so we will have some expression like numerator A(S) and denominator of S. So, we are supposed to get the denominator part, the denominator part, actually the characteristic equation. We are supposed to get the second order, but here we are getting a first-order equation. But why is this? So, it means there is one state that is not controllable. So, if the order is not 2, the rank is not full.

So, there is one problem: the system is not controllable. So, let us look how it is happening. Let us do the observation. So, now if you extend, if you split this equation,

$$\dot{X}_1 = -2X_1 + 2U$$

$$\dot{X}_2 = -X_2$$

So, if you notice here, this system, this system does not affected by the control input U. Because this U is going to affect the  $X_1$  and there is no direct link  $X_2$  and  $X_1$  so  $X_2$  is completely decoupled from the first equation so that's why this equation we can say this equation does not affect it by the complete input this is very very important takeaway so now if you want to get affected  $X_2$  dynamics so we need to modify  $X_2$  so if you have something instead of this one if you have for example system  $\dot{X}_2$  equal to minus  $X_2$  plus  $X_1$  then the system is controllable because here  $X_1$  is affected by bu and this  $X_1$  is going to affect  $X_2$  So, that is why this system, whole system is not controllable because  $X_2$  dynamics is decoupled from the control. So, this is how we can do the validation whether the system is controllable or not. So, there is a direct rule we can test the system because if you are asked to do the continuity analysis for a big system like the one we have, we had this aircraft system, this is quite difficult because we have to do lot of analysis.

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$$= \frac{1}{s^2 + 3s + 2} \begin{bmatrix} 3 & 2 \\ 0 \end{bmatrix} \begin{matrix} 2s+2 \\ 0 \end{matrix} \Rightarrow \frac{Y(s)}{U(s)} = C(SI - A)^{-1} B$$

$$= \frac{6s+6}{(s+1)(s+2)} = \frac{6}{(s+2)} = \frac{N(s)}{D(s)}$$

$$\dot{x}_1 = -2x_1 + 2u$$

$$\dot{x}_2 = -x_2 \leftarrow \text{does not affected by the control input.}$$

$$\dot{x}_2 = -x_2 + x_1$$

So, there is a direct rule we can use how to test. So, a linear system let me write, a linear system described by

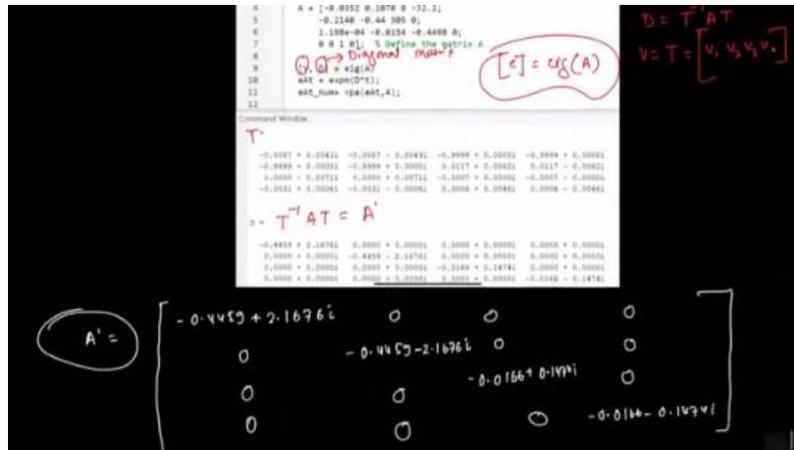
$$\dot{X} = AX + BU$$

b is controllable if and only if the contributing matrix which is

$$P = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

is rank n. So, it means if the if you assume here the A matrix which is n cross n square matrix and if the rank of this matrix is also 1, then we can say the system is controllable.

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There is another way also; you can test if the determinant of a matrix is not equal to 0. That is also another. So, this is the original in control synthesis; we do this criteria. Yes, the rank of this matrix is full, and then we can say the system is controllable. So, let us stop it here.

In the next lecture, we will continue; we will take an example of how we can test the controllability of the system. And we will also study how we start the control part. This is the main part in this course: how we can design the controller for a particular system once the system is fully controllable. Thank you.