

Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 55

Aircraft equations of motion for SIMULINK Block

Hello friends, last week we simulated pendulum and mass-spring damper systems in MATLAB and Simulink using different methodologies such as state space, transfer function, and non-linear methods. This week, we'll be simulating aircraft six-degree-of-freedom equations of motion in MATLAB and Simulink. Before we step onto that, it's a good starting point to simulate the first three-degree-of-freedom equations of motion. Now, let us go back to MATLAB. So, this is how it looks like: a three-degree-of-freedom block. So, this is a three-degree-of-freedom block in body axis. And these are the equations of motion. AXB, AZB, AXC, AZE, and so on and so forth. What we will do is derive these equations directly from the 6-DOF equations, then simulate them in MATLAB and Simulink, and then use the 3-DOF block. Now, let us go back to the 6-DOF equations of motion. Let me rewrite the equations, starting with the translational dynamics.

$$\begin{aligned}\dot{u} &= \frac{1}{m} [F_x^A + F_x^P + F_x^G] - qw + rv \\ \dot{v} &= \frac{1}{m} [F_y^A + F_y^P + F_y^G] - ru + pq \quad \dots Eq(15) \\ \dot{w} &= \frac{1}{m} [F_z^A + F_z^P + F_z^G] - pv + qu\end{aligned}$$

$$F_x^G = -mg \sin \theta, \quad F_y^G = mg \sin \phi \cos \theta, \quad F_z^G = mg \cos \phi \cos \theta$$

So again, I will not go into detail about how this equation has arrived, since that is not the intent of this course. Next is the rotational dynamics. We have 12 states, so there will be 12 equations.

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Translational Dynamics

$$\left. \begin{aligned} \dot{u} &= \frac{1}{m} \left[F_x^A + F_x^P + F_x^G \right] - g_w + x_v \\ \dot{v} &= \frac{1}{m} \left[F_y^A + F_y^P + F_y^G \right] - y_u + y_w \\ \dot{w} &= \frac{1}{m} \left[F_z^A + F_z^P + F_z^G \right] - z_v + z_u \end{aligned} \right\} \rightarrow Eq(15)$$

$$F_x^G = -mg \sin \theta \quad F_y^G = mg \sin \theta \cos \phi$$

$$F_z^G = mg \cos \theta$$

$$l = \dot{p}I_x - \dot{r}I_{xz} + qr(I_z - I_y) - I_{xz}pq$$

$$m_p = \dot{q}I_y + rp(I_x - I_z) + I_{xz}(p^2 - r^2) \dots Eq(16)$$

$$n = -\dot{p}I_{xz} + \dot{r}I_z + pq(I_y - I_x) + I_{xz}qr$$

Then we have rotational kinematics

$$\dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta$$

$$\dot{\theta} = q \cos \phi - r \sin \phi \dots Eq(17)$$

$$\dot{\psi} = q \sec \theta \sin \phi + r \sec \theta \cos \phi$$

Finally, we are left with translational kinematics, so that is written as

$$\begin{bmatrix} \dot{X}_e \\ \dot{Y}_e \\ \dot{Z}_e \end{bmatrix} =$$

$$\begin{bmatrix} \cos \psi \cos \theta & \cos \psi \sin \theta \sin \phi - \sin \psi \cos \phi & \cos \psi \sin \theta \cos \phi + \sin \psi \sin \phi \\ \sin \psi \cos \theta & \sin \psi \sin \theta \sin \phi + \cos \psi \cos \phi & \sin \psi \sin \theta \cos \phi - \cos \psi \sin \phi \\ -\sin \theta & \cos \theta \sin \phi & \cos \theta \cos \phi \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \dots Eq(18)$$

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Rotational Dynamics

$$\begin{aligned}
 I &= \dot{p} I_x - r I_{xz} + 2r(I_x - I_y) - I_{xz} p r \\
 m_y &= \dot{q} I_y + r p (I_x - I_z) + I_{xz} (p^2 - r^2) \\
 n &= -I_{xz} \dot{p} + I_z \dot{r} + 2p q (I_y - I_x) + I_{xz} 2r p
 \end{aligned}
 \left. \vphantom{\begin{aligned} I \\ m_y \\ n \end{aligned}} \right\} \rightarrow \text{Eq. (6)}$$

Rotational Kinematics

$$\begin{aligned}
 \dot{q} &= p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \\
 \dot{\theta} &= q \cos \phi - r \sin \phi \\
 \dot{\psi} &= q \sec \theta \sin \phi + r \sec \theta \cos \phi
 \end{aligned}
 \left. \vphantom{\begin{aligned} \dot{q} \\ \dot{\theta} \\ \dot{\psi} \end{aligned}} \right\} \rightarrow \text{Eq. (7)}$$

Now, the 3-DOF block considers the motion in the longitudinal xz plane. Here, we have

$$v = p = r = \phi = \psi = Y_e = 0$$

Hence, it greatly simplifies these equations. And finally, we can write the equations in this format:

$$\begin{aligned}
 \dot{u} &= \frac{1}{m} [F_x^A + F_x^P - mg \sin \theta] - qw \\
 \dot{w} &= \frac{1}{m} [F_z^A + F_z^P + mg \cos \phi \cos \theta] + qu \\
 \dot{q} &= \frac{m_y}{I_y} \\
 \dot{\theta} &= q \\
 \dot{X}_e &= u \cos \theta + w \sin \theta \\
 \dot{Z}_e &= -u \sin \theta + w \cos \theta
 \end{aligned}$$

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Translational Kinematics

$$\begin{bmatrix} \dot{x}_e \\ \dot{y}_e \\ \dot{z}_e \end{bmatrix} = \begin{bmatrix} c\psi c\theta & c\psi s\theta s\phi - s\psi c\phi & c\psi s\theta c\phi + s\psi s\phi \\ s\psi c\theta & s\psi s\theta s\phi + c\psi c\phi & s\psi s\theta c\phi - c\psi s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$\rightarrow E_p(18)$

$$v = \dot{\psi} = \dot{\theta} = \dot{\phi} = \dot{\gamma} = \dot{y}_e = 0$$

So, these are the reduced equations of motion in the longitudinal plane. So, we have the motion in the forward direction, that is u in the body x -axis, and then the motion is in w , that is the downward direction, and finally, we have the pitching moment about the y -axis. So here, we need to model these forces and moments that we are seeing over here. We need to model these forces as well as moments. We assume that thrust forces are acting through the aircraft CG in the body x -axis. So, we can represent propulsive forces as

$$F_x^P = T_{max} \delta_t$$

where T_{max} is the maximum engine thrust and δ_t varies from 0 to 1. So, 1 means we have maximum throttle, and if δ_t is 0, that means there is no engine thrust, that is 0. And we assume that. Propulsive forces in the z direction is 0.

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$$\left\{ \begin{array}{l} \dot{u} = \frac{1}{m} [F_x^A + F_x^P - m g \sin\theta] - g w \\ \dot{w} = \frac{1}{m} [F_z^A + F_z^P + m g \cos\theta] + g u \\ \dot{z} = m g / F_g \\ \dot{\theta} = q \\ x_e = u \cos\theta + w \sin\theta \\ z_e = -u \sin\theta + w \cos\theta \end{array} \right.$$

$T_{max} \rightarrow$ Max. Engine Thrust
 δ_t varies from 0 to 1

Now, aerodynamic forces are lift and drag, which act in the stability frame where the x-axis is aligned with the velocity vector. So, let me model these forces. So, we need to transform these forces to the body frame. So, aerodynamic forces are acting in the stability frame where the x-axis is aligned with the velocity vector. This is the angle of attack. This is the body x-axis. This is Z_b , and the lift vector will be perpendicular to the velocity vector. This is the lift, and drag will be parallel with the velocity but in the opposite direction. This is drag. If this is the angle of attack α , then this will also be α , and this angle will also be α .

So, what we will do is transform these lift forces to the body x-axis. Beginning with the lift force, we have here it will be $L \cos \alpha$, and here it will be $L \sin \alpha$. Similarly, resolving the forces, drag here will be $D \cos \alpha$, and here we have $D \sin \alpha$. So finally, we can write the forces in the x-direction—aerodynamic forces, to be specific—

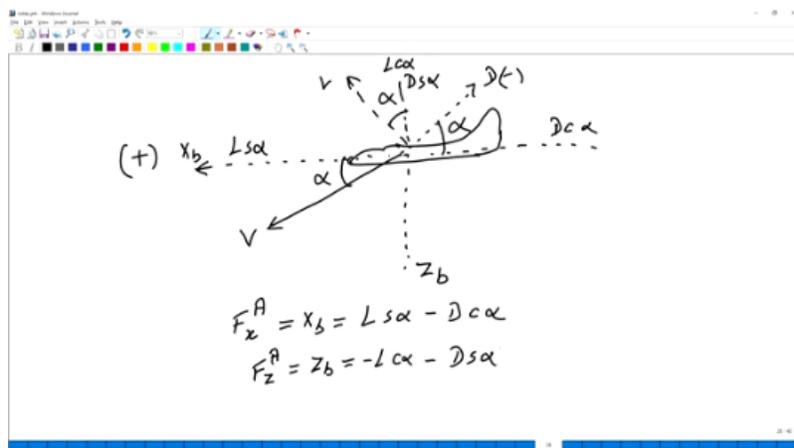
$$F_x^A = X_b = L \sin \alpha - D \cos \alpha$$

So, this direction is actually positive. And similarly, drag is acting in the opposite direction. So, this is negative. This direction is negative. This is positive. Similarly, we write

$$F_z^A = Z_b = -L \cos \alpha - D \sin \alpha$$

Now, how do we get this lift and drag?

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We know that lift is given by this expression

$$L = \frac{1}{2} \rho V^2 S C_L$$

$$D = \frac{1}{2} \rho V^2 S C_D$$

where ρ is the density, V is the total velocity, S is the wing area, C_L is the lift coefficient and C_D is drag coefficient. Now, we need to model this C_L and C_D .

$$C_L = C_{L0} + C_{L\alpha} \alpha + C_{L\delta e} \delta_e$$

$$C_D = C_{D0} + C_{D\alpha} \alpha + C_{D\delta e} \delta_e \quad \text{or} \quad C_D = C_{D0} + K C_L^2$$

Here, we are omitting q dependencies for simplicity. You can add as per your wish., what we are left with is pitching moment about the Y axis, the body Y axis. So, that means pitching moment. We know it is written as

$$m_y = \frac{1}{2} \rho V^2 S c C_m$$

$$C_m = C_{m0} + C_{m\alpha} \alpha + C_{m\delta e} \delta_e$$

Now, finally we can write

$$\dot{u} = \frac{1}{m} [L \sin \alpha - D \cos \alpha + T_{max} \delta_t] - g \sin \theta - qw$$

So, let me take this term as

$$A_{xe} = \frac{1}{m} [L \sin \alpha - D \cos \alpha + T_{max} \delta_t] - g \sin \theta$$

I am just aligning the nomenclature with what is given in the 3 dof block. What is A_{xe} ?

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Handwritten equations on a digital whiteboard:

$$L = \frac{1}{2} \rho V^2 S C_L \quad \text{and} \quad D = \frac{1}{2} \rho V^2 S C_D$$

$$C_L = C_{L0} + C_{L\alpha} \alpha + C_{L\delta e} \delta_e \quad \left[\begin{array}{l} \text{Omitting } q \text{ dependencies} \\ \text{for simplicity} \end{array} \right]$$

$$C_D = C_{D0} + C_{D\alpha} \alpha \quad \text{or} \quad C_D = C_{D0} + K C_L^2$$

$$m_y = \frac{1}{2} \rho V^2 S c C_m$$

$$C_m = C_{m0} + C_{m\alpha} \alpha + C_{m\delta e} \delta_e$$

A_{xe} is acceleration of body with respect to inertial reference frame so

$$A_{xe} = \frac{F_{x(A,P)}}{m} - g \sin \theta$$

$F_{x(A,P)}$ is nothing but aerodynamic or propulsive forces

$$F_{x(A,P)} = L \sin \alpha - D \cos \alpha + T_{max} \delta_t$$

now similarly we can write

$$\dot{w} = \frac{1}{m} [-L \cos \alpha - D \sin \alpha] + g \cos \theta + qu$$

$$\dot{w} = A_{ze} + qu$$

$$A_{ze} = \frac{F_{z(A,P)}}{m} + g \cos \theta$$

$$F_{z(A,P)} = -L \cos \alpha - D \sin \alpha$$

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The image shows a whiteboard with handwritten mathematical derivations. At the top, it shows the equation for \dot{u} with a bracket under the aerodynamic and propulsive force terms labeled A_{xe} . Below this, it defines A_{xe} as the acceleration of the body with respect to the inertial reference frame. The next line shows $A_{xe} = \frac{1}{m} [L \sin \alpha - D \cos \alpha + T_{max} \delta_t] - g \sin \theta$. This is followed by $A_{xe} = \frac{F_{x(A,P)}}{m} - g \sin \theta$. Then, it shows the equation for \dot{w} with a bracket under the aerodynamic and propulsive force terms labeled A_{ze} . Finally, it shows $\dot{w} = A_{ze} + qu$ and $A_{ze} = \frac{1}{m} [-L \cos \alpha - D \sin \alpha] + g \cos \theta + qu$.

I am writing in this format because we will be using this in our simulation. Now finally writing three degree of freedom equations as

$$A_{xb} = \dot{u} = A_{xe} - qw$$

$$A_{zb} = \dot{w} = A_{ze} + qu$$

$$\dot{\theta} = q$$

$$\dot{q} = \frac{m_y}{I_y} \quad \dots Eq(19)$$

$$\dot{X}_e = u \cos \theta + w \sin \theta$$

$$\dot{Z}_e = -u \sin \theta + w \cos \theta$$

So, where A_{xb} and A_{zb} are the accelerations of body with respect to body fixed coordinate frame now let us check these equations directly in the matlab 3dof block

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Handwritten equations on a whiteboard:

$$A_{ze} = \frac{F_z(A,P)}{m} + g \cos \theta \quad \left\{ \begin{array}{l} f_{z(b)} = -L \cos \theta - D \sin \theta \end{array} \right.$$

finally, writing 3Dof Equations as

$$\left. \begin{array}{l} A_{xb} = \dot{u} = A_{xe} - g w \\ A_{zb} = \dot{w} = A_{ze} + g u \\ \dot{\theta} = q \\ q = m_q / I_y \\ \dot{x}_e = u \cos \theta + w \sin \theta \\ \dot{z}_e = -u \sin \theta + w \cos \theta \end{array} \right\} \text{Eq. (17)}$$

$A_{xb}, A_{zb} \rightarrow$ Accelerations of body w.r.t. b

All right. Now, before I proceed further, I have seen students struggle with the units of these equations, whether they have to use in the numericals or in Simulink. So I always suggest to match the units on the left hand sides and the right hand side. So, for example, A_{xb} or \dot{u} . Here \dot{u} this is in meters per second square. So, \dot{u} unit should be meters per second square. So, in \dot{u} what do we have? We have $A_{xe} - qw$. So, here q is the rate that is radians per second and w is what that is the velocity meters per second. So, we get radians meters per second square and we know the radian is some angle upon angle which is unit less. So, finally, we have meters per second square. So, this qw is meters per second square. So, you have to ensure that q is not in the degrees per second, it should be in radians per second only. All right. And MATLAB by default, it takes the values in radians. So we have to be absolutely sure what unit we are using. Now coming to A_{xe} . Now what is A_{xe} ?

$$A_{xe} = \frac{F_{x(A,P)}}{m} - g \sin \theta$$

So here g unit is meters per second squared. Here θ is in radians. So we get meters per second square which is all right. Now if we consider force in newtons and the mass is in kgs. this can be further simplified as kilogram meters per second square divided by

kilogram so finally we have meters per second square, so ensure this force is always in newtons

$$F_{x(A,P)} = L \sin \alpha - D \cos \alpha + T_{max} \delta_t$$

δ_t is unitless; it is 0 to 1. And if T_{max} is given in kgs, convert it to newtons, so we have this in newtons. Similarly, this angle should be in radians, and this drag should be in newtons. This L should be in newtons. Now, what is L? L is the lift, which is given as

$$L = \frac{1}{2} \rho V^2 S C_L$$

So, this L's unit should be in newtons.

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Handwritten notes on a whiteboard showing unit derivations for acceleration and force. The notes include:

- $\dot{v} \rightarrow m/s^2$
- $\dot{v} = a_{xe} - g_w$
- $a_{xe} = \frac{F_x(a,p)}{m}$
- $g \sin \alpha$ (rad) m/s^2
- $\frac{N}{kg} = \frac{kg \cdot m/s^2}{kg} = m/s^2$
- $F_x(a,p) = L \sin \alpha - D \cos \alpha + T_{max} \delta_t$ with units (N) and (kg) indicated.

So, density is given as kilogram per meter cube. Then we have velocity meters per second or meter square per second square, and area is in meter square. So we have kilogram, this is meter power 4, this is meter cube, so we have kilogram meters per second square, which is nothing but newtons. All right, so lift should be force should be in newtons. Then we have $\dot{\theta} = q$, so this is radians per second. It should be radians per second, and this is rate. It's already radians per second; it should not be in degrees per second. All right, then we have moment about y axis

$$m_y = \frac{1}{2} \rho V^2 S c C_m$$

So, q unit is radians per second. This is \dot{q} ; it should be radians per second square or it should be simply 1 upon second square. Now, let us see what is the unit of m y in this case. Again, density is in kilogram per meter cube, velocity is meters per second, so we

have meter square per second square. Area is in meter square. C_m doesn't have any unit. c is chord, which is given in meters, so we have kilogram meter square per second square. What about. Moment of inertia, the unit of moment of inertia is kilogram meter square. So, finally, we have

$$\frac{m_y}{I_y}$$

m_y we got kilogram meter square per second square, and I_y is kilogram Square this. Finally, we have 1 upon second square, so we have 1 upon second square, and this is \dot{q} is 1 upon second square. All right. Now, the final one on the left is \dot{X}_e which is given as

$$\dot{X}_e = u \cos \theta + w \sin \theta$$

So, this unit should be meters per second. So, velocity. So, this angle will be in radians. Angle is in radians. And this is again the unit of meters per second. Alright, so similarly, you can find it for \dot{Z}_e . The unit will also be in meters per second. So, now I hope there is no confusion in the units. Now, further, we need some aircraft data for this simulation.

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The image shows handwritten mathematical derivations for unit analysis. The equations are as follows:

- $l = \frac{1}{2} \rho v^2 S C_L = \frac{\text{kg}}{\text{m}^3} \frac{\text{m}^2}{\text{s}^2} \text{m}^2 = \frac{\text{kg} \cdot \text{m}}{\text{s}^2} = \text{N} \checkmark$
- $\theta = \frac{q}{v} = \frac{\text{rad}}{\text{s}} \cdot \frac{\text{s}}{\text{m/s}} = \text{rad} \checkmark$
- $q = \frac{m_y}{I_y} \quad \frac{\text{rad}}{\text{s}^2} = \frac{1}{\text{s}^2} \checkmark$
- $m_y = \frac{1}{2} \rho v^2 S C_m c = \frac{\text{kg}}{\text{m}^3} \frac{\text{m}^2}{\text{s}^2} \text{m}^2 \text{m} = \frac{\text{kg} \cdot \text{m}^2}{\text{s}^2} \checkmark$
- $I_y \rightarrow \text{kg} \cdot \text{m}^2$
- $\frac{m_y}{I_y} = \frac{\frac{\text{kg} \cdot \text{m}^2}{\text{s}^2}}{\text{kg} \cdot \text{m}^2} = \frac{1}{\text{s}^2} \checkmark$
- $\dot{X}_e = v \cos \theta + w \sin \theta$
(rad)

So, let us stop it here. We will begin with the aircraft data and the simulation in the next lecture. Thank you.