Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 51

Second order system and its solution

Hello friends, in the last lecture, we discussed the first-order systems, the significance of a time constant, and the steady-state gain. We also started discussing the second-order systems. So, this was the second-order system that we had written here, which we represented as equation number 4. I missed writing equation number 2; this equation is actually equation number 3, sorry. Now, let us continue where we stopped. So, we are aware of this equation: linear second-order systems can be represented by

$$\frac{d^2}{dt^2}y(t) + 2\xi\omega_n\frac{d}{dt}y(t) + \omega_n^2\, y(t) = K\omega_n^2\, u(t) \quad \ldots Eq(4)$$

Here, we know that ξ represents the damping ratio, and ω_n is the undamped natural frequency. We are interested in finding the unit step response. The unit step response of a second-order system is the solution of y in equation number 4 when y(0) = 0 and u(t) = 1 for t ≥ 0. We know that the unit step response assumes three forms, depending on the damping ratio: greater than 1, between 0 and 1, or less than 0, based on the location of the algebraic equation. The algebraic equation can be represented as

$$S^2 + 2\xi\omega_n S + \omega_n^2 = 0 \quad \ldots Eq(5)$$

Again, this has been discussed by Professor Dipak Giri in the first course. Please revise your concepts if you are unfamiliar with these equations. This equation is generally known as the characteristic equation. We need these equations so that we can use them in our MATLAB environment. Hence, we are revising this. And the roots are:

$$S_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \quad \ldots Eq(6)$$

Here, the frequency of oscillation is represented by this variable $\omega_n\sqrt{1-\xi^2}$, which we call the damped natural frequency. This is generally represented by ω_d.

$$\omega_d = \omega_n\sqrt{1-\xi^2} \quad \ldots Eq(7)$$

Now, we know that the frequencies ω_d and ω_n have meaning only for the underdamped case. For the underdamped case, we know that ξ is 0 less than ξ less than 1. So in this case, ω_d is generally less than ω_n. That is, the damped frequency is always less than the undamped frequency. Now, we know that most practical applications deal with the underdamped case, so our focus will only be on the underdamped case. Let me write the solution of the unit step response for the underdamped case.

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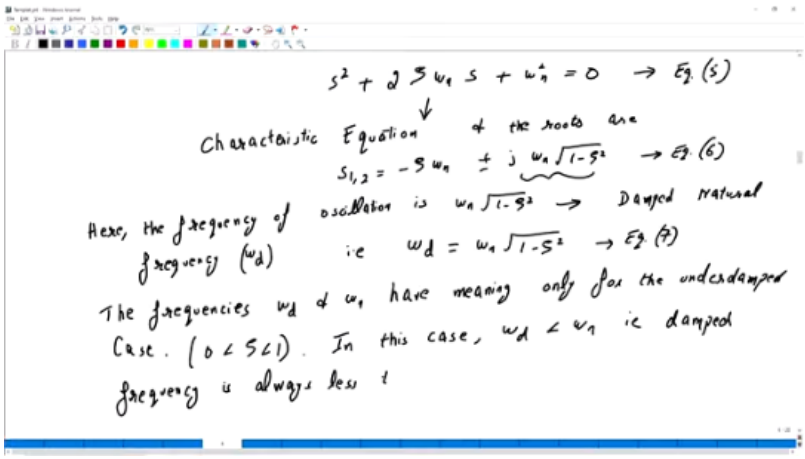

We will be plotting these responses in MATLAB. So, the unit step response of the underdamped case is given as (I am directly writing the final expression): Again, this has been extensively covered in the first part of this course.

$$y(t) = K\left[1 - e^{-\xi\omega_n t}\left\{\cos\omega_d t + \frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right\}\right] \qquad for\ t \geq 0 \quad \ldots Eq(8)$$

We can also represent in terms of the phase angle in terms of phase angle. In terms of ϕ phase angle ϕ and when the input is constant that is $u(t) = A$ then equation 8 can also be written as in terms of phase angle

$$y(t) = AK\left[1 - \frac{\omega_n}{\omega_d}e^{-\xi\omega_n t}\sin(\omega_d t + \phi)\right] \qquad for\ t \geq 0 \quad \ldots Eq(9)$$

$$\phi = tan^{-1}\frac{\omega_d}{\xi\omega_n} = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$$

Now, we will use this equation for the second order system. We know that this mass spring damper system is represented by the second order.

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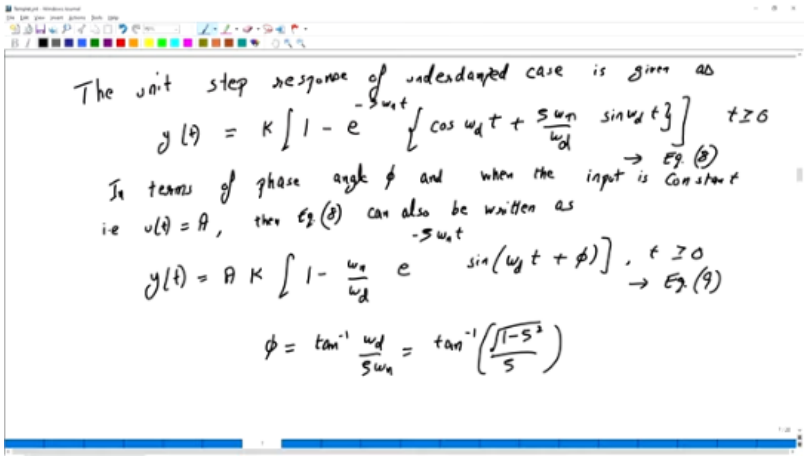

Let me draw the diagram here. This is the force f. This is the displacement represented by y. This is damper c. This is k_s. So we know that equations of motion for mass spring damper system. is represented as

$$m\frac{d^2}{dt^2}y(t) + c\frac{d}{dt}y(t) + k_s y(t) = f(t) \quad \ldots Eq(10)$$

here k_s is written just to make it little bit different from the steady state gain k just to avoid confusion, now here y of t is represented as displacement of block of mass m from its static equilibrium position f of t is nothing but the input force. k_s represents the stiffness of the spring. And c is the damper. So now, the system can also be written in the form of equation number 4. That is,

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k_s}{m}y = \frac{1}{m}f \quad \ldots Eq(11)$$

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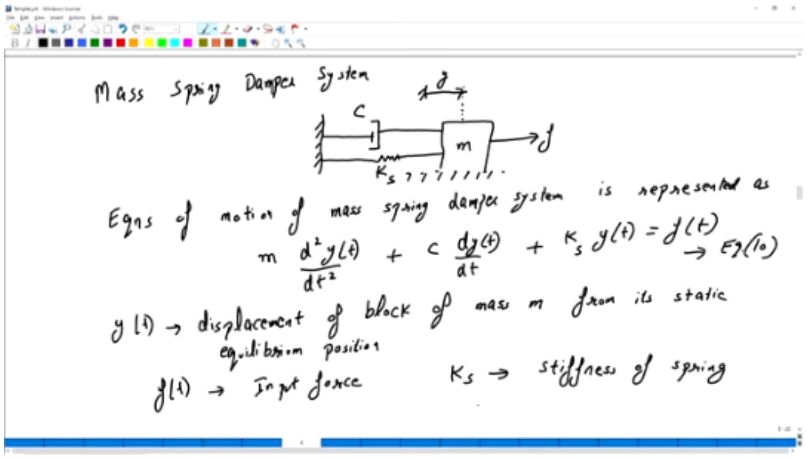

Again, I am dropping the t term for convenience. Now, if you compare this equation with the characteristic equation, we can easily get

$$2\xi\omega_n = \frac{c}{m}$$

$$\omega_n^2 = \frac{k_s}{m}$$

$$K\omega_n^2 = \frac{1}{m}$$

Now, finally, solving for parameters K, ω_n and ξ yields

$$\omega_n = \sqrt{\frac{k_s}{m}}$$

$$\xi = \frac{c}{2\sqrt{k_s m}}$$

$$K = \frac{1}{k_s}$$

Now, we can easily obtain the transfer function. For equation number 11, this equation number 11. We will be considering taking the Laplace transform of equation number 11 with zero initial conditions. Now, we always consider zero initial conditions whenever we take the Laplace transform. So, this can be represented as

$$S^2Y(S) + \frac{c}{m}SY(S) + \frac{k_s}{m}Y(S) = \frac{1}{m}F(S)$$

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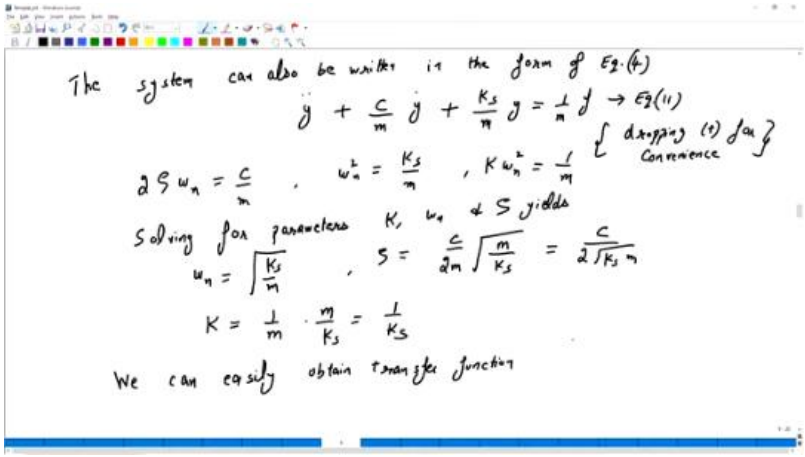

I can take Y of S common from LHS.

$$T.F. = \frac{Y(S)}{F(S)} = \frac{1}{mS^2 + cS + K_s} \quad \ldots Eq(12)$$

So, this is the representation for the transfer function. Now, with this, we are ready to switch to MATLAB and implement this transfer function—equation number 12—and plot the position and velocity using different methodologies. So, in the next lecture, we will be simulating this in position and velocity in the MATLAB environment.

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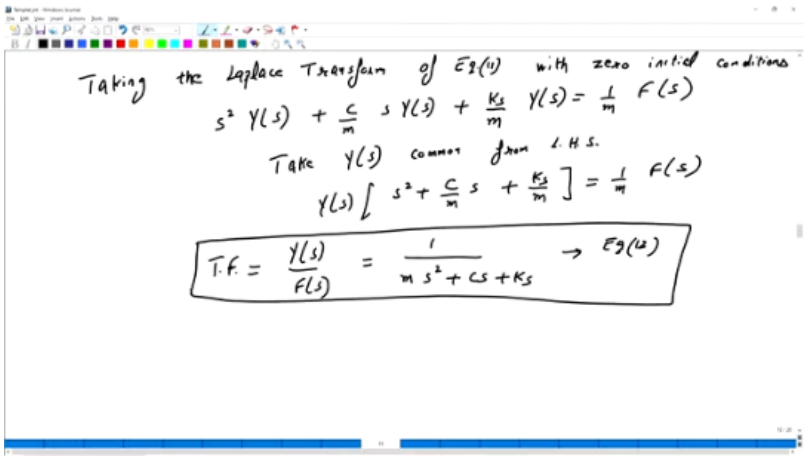

Thank you.