

Advanced Aircraft Control Systems With MATLAB / Simulink

Prof. Dipak K. Giri

Department of Aerospace Engineering

Indian Institute of Technology Kanpur

Lecture 05

Linear Transformation of States

Hello everyone, welcome to the lecture. In today lecture we will be discussing how we can convert the original system which is may be in the coupled form to a decoupled form, so that we can easily check whether the system is stable or not, by looking at diagonal elements of the matrix. So, this concept we call linear transformation. We can or we will map the system from one domain to another domain, using some transformation matrix and that transform system, we call sometimes canonical form. So, let us start the lecture and I hope you understand some important part with this lecture.

So, we are going to cover the contents. First is state transformation It is also called diagonalization approach. And also we'll discuss the eigenvalues. and the eigenvectors and how they will these are the important parameters to come up with.

So, if you have a system, for example, if you have a system,

$$\dot{x} = ax, x(0) = 4$$

So, if you solve this equation, we can write

$$x(t) = e^{at}x(0) = 4e^{at}$$

so it means if you have a system in the structure in this structure it is easy for us to determine or comment, whether the system is stable or not because the here \dot{x} , there is no other parameters involved in the system states the system which is taking, which is the derivative of the state, the same state available at the right hand side. So, here it is easy to show and easy to comment how the system is going to behave based on the polarity of the A matrix, sorry, A, this is the scalar matrix.

So, if the, if A is, for example, if A is negative, we can comment that. So, we can say

$$x(t) = e^{-at}4$$

So, the system will be something like this. It will start from 4 and be exponentially decaying to 0.

And so, based on the polarity of A, we can comment whether the system is stable or not. The same approach will follow in today lecture. So, how can come up with the some diagonal form of the original system and it will be easy for us to comment whether the system is, how the system is going to behave about stability. So, our main motivation is suppose we have system, we have system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

we have now, if you find some way which can help us to convert the system in some other domain, maybe

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{bmatrix} -z_1(t) \\ -z_2(t) \end{bmatrix}$$

So, by the polarity of values of c_1 , c_2 we can comment the system is stable or not. so here this process is called linear transformation, so here we'll be using some matrix which is actually linear transformation, matrix transformation which will help us to solve this problem and this is also sometimes we call canonical form, so let's start, also here we have the original system, for example we have the system which may be in the coupled form,

$$\dot{X} = AX + BU$$

so here we are assuming A is not diagonal diagonal matrix, okay so here we are introducing a new variable z here, z is the new variable we are defining new variable and this new variable is related by the following equality. So, we are writing

$$X(t) = TZ(t)$$

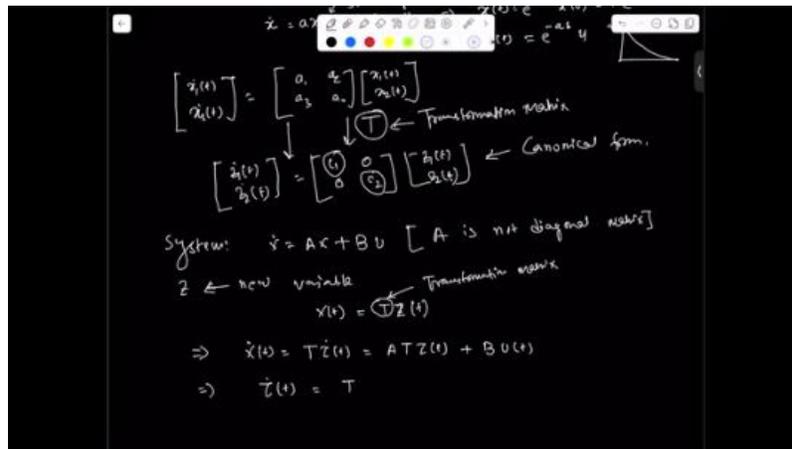
So, here using this transformation matrix which help us to transform x domain to z domain. And now, so this is basically we call this matrix as transformation matrix. Now, if you substitute this expression or this equation, the original state equation we can write, we can get

$$\dot{X}(t) = ATZ(t) + BU(t)$$

$$\dot{Z} = T^{-1}ATZ(t) + T^{-1}BU(t) \dots Eq(1)$$

so let us define $A' = T^{-1}AT$ and $B' = T^{-1}B$

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So, if we substitute these parameters, this parameter in equation one we can write,

$$\dot{Z} = A'Z(t) + B'U(t) \dots Eq(2)$$

and the output equation, we can write,

$$Y(t) = CX(t) = CTZ(t) = C'Z(t)$$

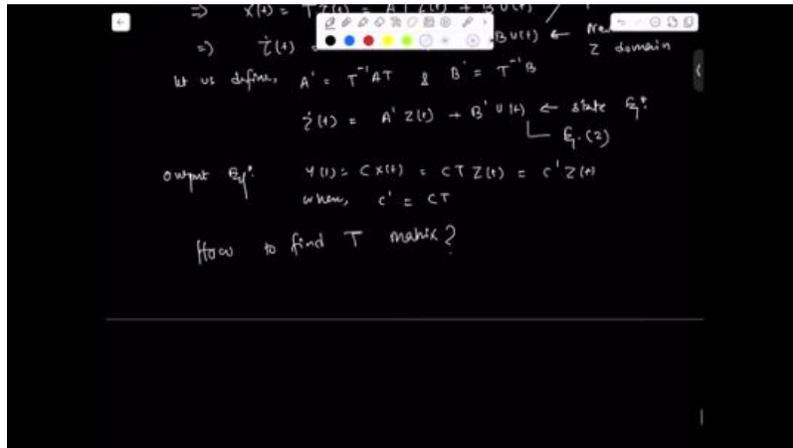
Where $C' = CT$. So, we are having two equations, one is the state equation and this is the output case, the output equation and this is the state equation, right. And now the question is how to find this T matrix, but before that here I should comment that the system which is, which you have found here, here actually this A dash is actually diagonalized. Now how is it coming? we will explain. So, this is the diagonal system.

So, by looking at the diagonal elements of A' we can comment how this J z a parameter and still going to behave over time. This is, we can easily comment. You see, this is the z_1, z_2, z_{31} , this z vector and \dot{z} and we have the diagonal element. So, the system is completely decoupled.

So, if the states are decoupled from each other, we can easily find the solution. So, that is why it is very very powerful technique if you can convert the original system which is in the coupled form into decoupled form. So, now the question is how to find T matrix. So, now we will go to some linear algebra concept to find this T matrix.

So, here The T matrix, first we have to find the eigenvector. The T matrix actually, T matrix can be formed, can be formed by the eigenvectors. So, we can say T is nothing but if there is 2 cross, suppose for example, we have that we are using a matrix 2 cross 2. So, we will have 2 eigenvalues.

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For the corresponding eigenvalues, we will have eigenvector, right. So, for this 2 cross 2 matrix, we will have 2 eigenvectors, which will be

$$T = [v_1 \quad v_2]$$

So, this is nothing but the transformation matrix. Okay. Now, the question is how to find eigenvalues?

Because eigenvalues will help us to find the eigenvector. So, eigenvalues, if you remember in our earlier lecture, we have used

$$|\lambda I - A| = 0$$

So, if you, these are the eigenvalues, this is lambda actually, eigenvalues. So, if you solve this equation, we will get eigen values of A. So, this is also sometime called as characteristic equation. For a particular system, remember the characteristic equation is always fixed. the particular system because we are getting the A system matrix here. So, all these characteristic equation fixed for a specific system. Now, this characteristic equation if we expand, we will have in this form. We can, if there are, if you are using A is n cross n matrix, the order of the characteristic equation will be also n.

$$\lambda^n + a_n \lambda^{n-1} + a_{n-1} \lambda^{n-2} + \dots = 0$$

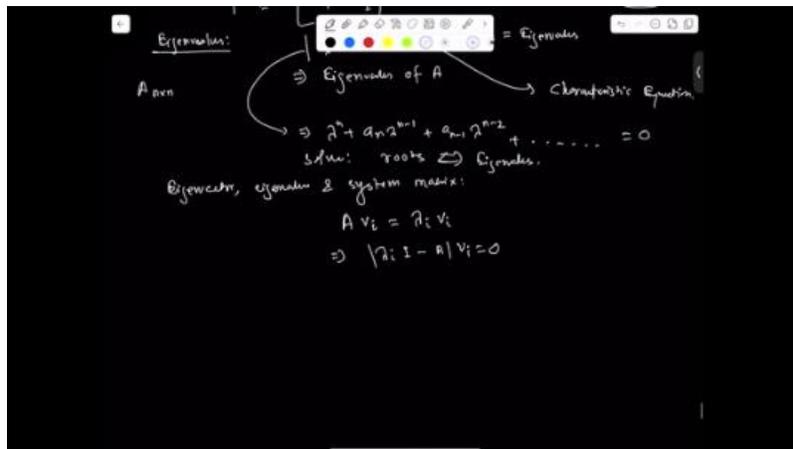
So, it will move like this equal to 0. So, if you solve this equation, it will give the list of eigenvalues or we can say the roots. Basically, if we solve, we will get the roots roots these are nothing but eigenvalues okay, now if you know the eigenvalues, if you have the eigenvalues and if you have the a matrix which is here, then easily we can find the corresponding eigenvector, so as you know the eigenvector and eigenvalues satisfy the following equation So, eigenvector, eigenvalues and system matrix satisfy the following equation,

$$Av_i = \lambda_i v_i$$

$$|\lambda_i I - A| v_i = 0$$

So, from this equation we can find because here the only unknown because from this equation, from this equation we will find the eigenvalues and so in this equation the only unknown is the eigenvector because a is already given to us. So, now once you have, this is very important note, when the system has distinct eigenvalues, we can find the system into a canonical form, where the state equation will be in decoupled form

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We will take an example later in this lecture, then we will come to know. So, finally, our T, the transformation matrix, which is nothing but T, can be formed if there are n eigenvalues. So, those eigen, if there are n eigenvectors, these will come into the, as a column in the T matrix. this is how we can form the eigen, this is how we can form the transformation matrix.

$$T = [v_1 \quad v_2 \quad \dots \quad v_n]$$

Now, we will go back to our system matrix in the transform domain. So, this is what we have found in our lecture,

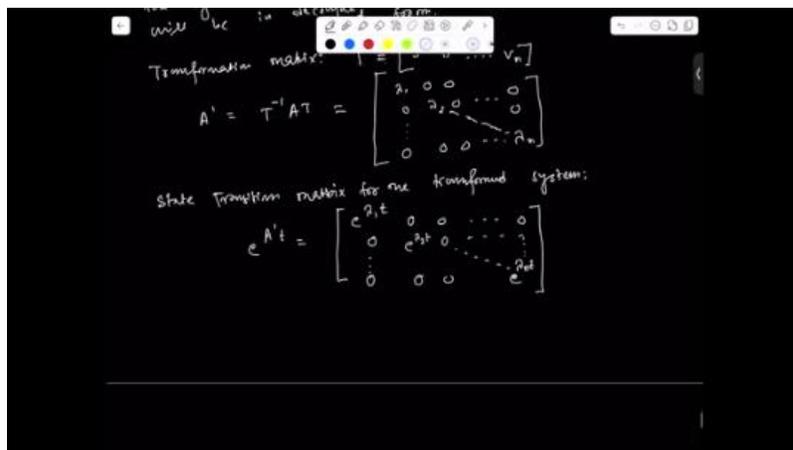
$$A' = T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

in the diagonal we will have the eigenvalues. So, this is how we can find in the decoupled form of the system. So, only if you notice that in the A dash matrix, we are having only the diagonal elements which contains the eigenvalues of the system. now in this case if you want to write the state transition matrix, if you would like to write the state transition matrix state transition matrix for the transformed system, we can write

$$e^{A't} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_n t} \end{bmatrix}$$

So, we will get the system transition matrix in structure. Now, we will find the relation between, how can we find the relation transform system matrix using the state transition matrix and linear transformation matrix.

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So, let us look So if you extend our transform matrix A' using the power series form, in power series, we can write, into our A' term, we can write, Similar to our original system matrix, the way we wrote for the original system matrix in power series, original state structure matrix. For the original system, we had I plus A dash, the same structure we had, but here the matrix, system matrix will be different from the transform system,

$$\begin{aligned}
e^{A't} &= I + A't + \frac{A'^2 t^2}{2!} + \frac{A'^3 t^3}{3!} + \dots \\
&= T^{-1} \left[I + At + \frac{A^2 t^2}{2!} + \dots \right] T \\
e^{A't} &= T^{-1} A T \\
E^{At} &= T e^{A't} T^{-1}
\end{aligned}$$

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Handwritten derivation showing the relationship between the state transition matrix in the original system and the transformed system:

$$\begin{aligned}
e^{A't} &= I + A't + \frac{A'^2 t^2}{2!} + \frac{A'^3 t^3}{3!} + \dots \\
&= T^{-1} T + T^{-1} A T t + \frac{(T^{-1} A T)(T^{-1} A T) t^2}{2!} + \frac{(T^{-1} A T)(T^{-1} A T)(T^{-1} A T) t^3}{3!} + \dots \\
&= T^{-1} T + T^{-1} A T t + T^{-1} A^2 T \frac{t^2}{2!} + T^{-1} A^3 T \frac{t^3}{3!} + \dots \\
&= T^{-1} \left[I + A t + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots \right] T \\
\Rightarrow e^{A't} &= T^{-1} e^{At} T \\
\Rightarrow e^{At} &= T e^{A't} T^{-1}
\end{aligned}$$

This is very, very powerful relation. how the original system and transformed system, state transition matrix are related to each other. So, now we will find the solution of the state equation, if they are written in the transformed form because we found the solution of the system which are written in the original state-based form. But also in this case also you can find the solution of the transform system, right. So, now in this original system we had before solution of

$$\begin{aligned}
X(t) &= e^{At} X(0) + \int_0^t e^{A(t-\tau)} B U(\tau) d\tau \\
Z(t) &= e^{A't} Z(0) + \int_0^t e^{A(t-\tau)} T^{-1} B U(\tau) d\tau
\end{aligned}$$

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$$\Rightarrow e^{At} = T^{-1} e^{A^d t} T$$

$$\Rightarrow e^{At} = T e^{A^d t} T^{-1}$$

The solution of $x(t)$: $x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$

$$\Rightarrow T z(t) = e^{A^d t} T z(0) + \int_0^t e^{A^d(t-\tau)} B u(\tau) d\tau$$

$$\Rightarrow z(t) = T^{-1} e^{A^d t} T z(0) + T^{-1} \int_0^t T^{-1} B u(\tau) d\tau$$

So, this is the solution of the transform system. And if the anyways here if you consider unforced system or uncontrolled system, if you consider uncontrolled system, so this part is going to be 0. So, we will have this equation. So, if you write in the system, it is not controlled or is not affected by the external inputs.

So, we have the solution,

$$Z(t) = e^{A^d t} Z(0)$$

And now, if you would like to solve the original system solution, so it is very, quite interesting. So, here we have found the expression of $Z(t)$ in transform domain. Now, if you want to get back to the original system solution, then how can you find? It is squared.

So, what do we know? We know

$$X(t) = T Z(t) = T e^{A^d t} Z(0)$$

So, this is the solution, also we find the original system solution. So, this is very, very important, how we can find the diagonalized system and also how we can find the solution of the diagonalized system and how eigenvector and eigenvalues are very important parameters for finding the linear transformed matrix.

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$Z(s) = T^{-1} B U(s)$

↳ sys. if the decoupled system.

If our system is not coupled or does not affect by external input.

$Z(t) = e^{A^* t} Z(0)$

$x(t) = T Z(t) = T e^{A^* t} Z(0)$

So let's stop it here. We'll continue from the next lecture where we'll have example on this particular topic. Thank you.