Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 44

Introduction to Sliding Mode Control

Hello everyone, now we are going to start a new topic: sliding mode control. In this topic, we will start with the main concept of this approach: how we can design nonlinear control for any nonlinear system. So before we proceed to the main mathematical design concept, first let me come up with some definitions, which will be very important for understanding the why we have to consider sliding mode control for control synthesis. For any practical control problem, there will always be a discrepancy between the actual plant and its mathematical used for the controller design. These discrepancies arise from external disturbances, plant parameters, and unmodeled dynamics. Designing effective control laws in the presence of these disturbances or uncertainties is a very challenging task for control engineers. This led to the development of robust control robust control methods, and one such method is the sliding mode control technique So here, the sliding mode control is a robust control technique. When the system is not properly known due to unknown external disturbances, plant parameters, and unmodeled dynamics, how can we come up with an effective control that will provide a robust control algorithm or law? So now We'll take a simple example and try to understand how we can design sliding mode control effectively for a nonlinear system. So first, let us understand. Let's take a nonlinear system. So let us consider, let us understand the sliding mode control control through the following example so we have the system for example second order system

$$\ddot{x} = f + u \quad \ldots Eq(1)$$

so here f is the unknown parameters no this is the system parameters function and u is the control so here um we can assume this f can be disturbance or some unknown forces acting on the system and u is the control into the system so we can come up with this kind of system in our daily life so now we can write in state space. If we write in state space equation one, we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = f + u \quad \dots Eq(2)$$

x1 dot equal to x2 and x2 dot equal to a plus u. So this is 2 equation and here x_1 we can assume as, let us assume this is the position and x_2 is the velocity. Let us assume. And here f is disturbance or unknown force. Okay. So also let us assume here this f is bounded.

$$|f| \leq \Delta > 0$$

so it means if less than some value which is greater than zero some norm for the function f so our control objective is objective so in this example we are going to devote two lectures how we can design sliding mode control for this particular system if you can understand this concept the mathematical design concept for this equation number one the same concept can be applied for the other dynamical system so here our control objective is control objective design control law which is u such that it drives x_1 and x_2 to origin asymptotically.

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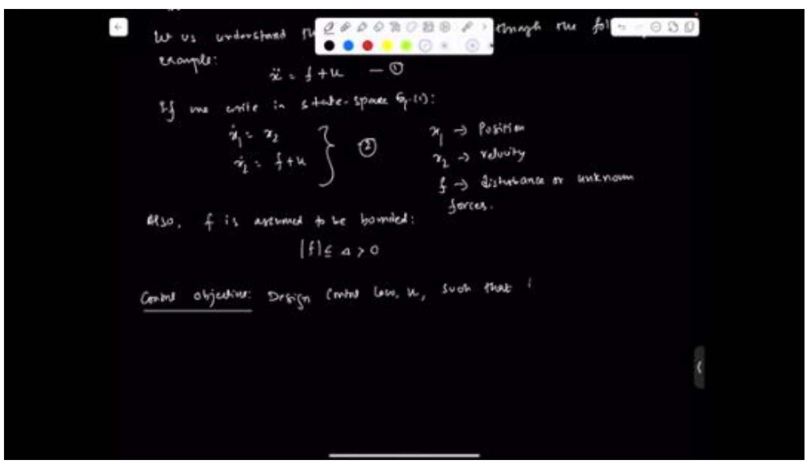

That is, we can write as limit t tends to infinity x_1 and x_2 goes to zero so this is our mission objective and we will fulfill this design objective using sliding mode control but to achieve this for us if this asymptotic convergence of x_1 and x_2 is a challenging problem and so we can simply state we can if you want to design simply state feedback control we can do it right in the absence of this non-linear function f so if you consider if you simply use state feedback control state feedback control in the absence of absence of f which is unknown function we can easily derive the control algorithm for this system defined by equation two so in the absence of this f equation equation 2 can be written as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

and if you consider state feedback control

$$u = -k_1 x_1 - k_2 x_2$$

where k_1 and k_2 are positive gain then our this system yields to be

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -k_1 x_1 - k_2 x_2$$

so this is basically closed loop system this concept already you have covered while in the modern control closed loop system and if you plot uh if you do the MATLAB plot for this particular system now for the initial condition for example $x_1(0) = 1$ and $x_2(0) = -2$ and if you consider $k_1 = 3$ and $k_1 = 4$ so this is so we'll have the result something like this something like this right which is maybe this is for example minus 2 1 and this is 0 so this is basically x_1 and x_2 so this is basically x_1 this is x_2 and so this result we obtain when f equal to zero so we can do it in similar uh you can try this MATLAB code yourself this is a very easy MATLAB code how to do this now um you can see that asymptotic convergence achieved for this particular case so here basically we can say that asymptotic convergence is achieved for x_1 and x_2 now how to tackle this system once we have this unknown function f because in this for this MATLAB code we have not considered if f in the system the non-linear function now we'll consider the non-linear function and we'll see the result again so now

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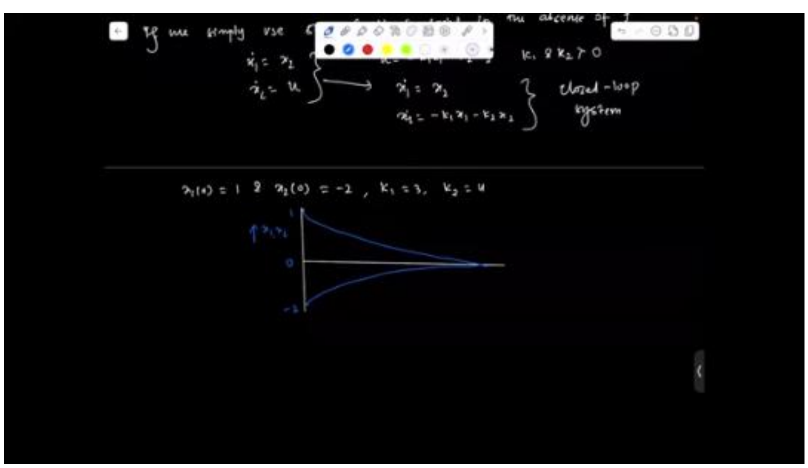

let us say when $f = \sin t$ for example this is the nonlinear function sine of t is a nonlinear function and we are going to consider this function in the system so let's assume this is the disturbance as a disturbance for the same for the same initial condition initial conditions and control gain we have the simulation results this is a result so the previous code also you can design using this MATLAB code so if you consider here we have considered f here in this particular case you have considered f but if you don't consider f we'll get result something like this something like this so when you are considering a in our system and if you use state feedback control our system response are not asymptotic convergence so it is fluctuating over time so we have the source oscillations in the system response now let's assume let's assume this is the upper bound of this this oscillation defined by omega let's assume so here we can say we can write here some notes for f not equal to zero the linear feedback because here we are using linear feedback if you notice here we are using linear feedback control what you have done in the modern control so if you use linear control for f not equal to zero

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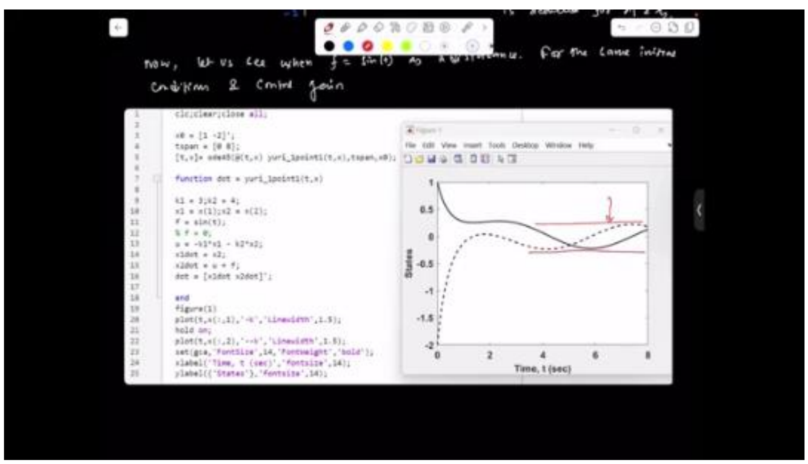

Because we have taken f equal to sine of T, the linear feedback laws only drive the states to the bounded domain. That is clear. This bounded domain is defined by $\Omega(k_1, k_2)$ This is so we can define it like this, and for f, this is our Δ, some value, some normal. Okay, so to tackle this problem, this is how we can come up with some control synthesis. If there is a bounded disturbance, how can we create asymptotic convergence of the system states? This is the main motivation for sliding mode control. So, how can we tackle these issues? In the presence of this bounded disturbance, how can we tackle this problem? So now, we'll go step by step to solve these issues. Let us consider or we can write a good candidate. A candidate would be homogeneous linear state, or we can write a linear time-invariant differential equation. We are choosing some system, a linear system which is time-invariant using the state variables x_1 and x_2. So here, we have chosen some equation:

$$\dot{x}_1 + kx_1 = 0$$

$$\dot{x}_1 = -kx_1 \quad ... Eq(3)$$

Where $k > 0$. So, this is the homogeneous linear time-invariant function we are assuming. And our main motivation is how we can drive our states to follow this trajectory. This is the main motivation of sliding mode control in this lecture. So now, if you get the solution of this. The solution of equation 3 would be

$$x_1(t) = x_1(0)e^{-kt}$$

And since

$$x_2 = \dot{x}_1 = x_1(0)e^{-kt} = -kx_1(0)e^{-kt}$$

Okay, so we can notice that both x_1 and x_1 converge to 0 asymptotically. Okay, we Also, we can notice if you have the system, some this system, this homogeneous linear time unit system, if you have system like this, so there is no effect of disturbance also in this equation 3, right? So, we can achieve this by introducing a new variable. in the state space of the system of equation 1. So, what we are going to do is, we are introducing a new variable. Let us introduce a new variable in the form

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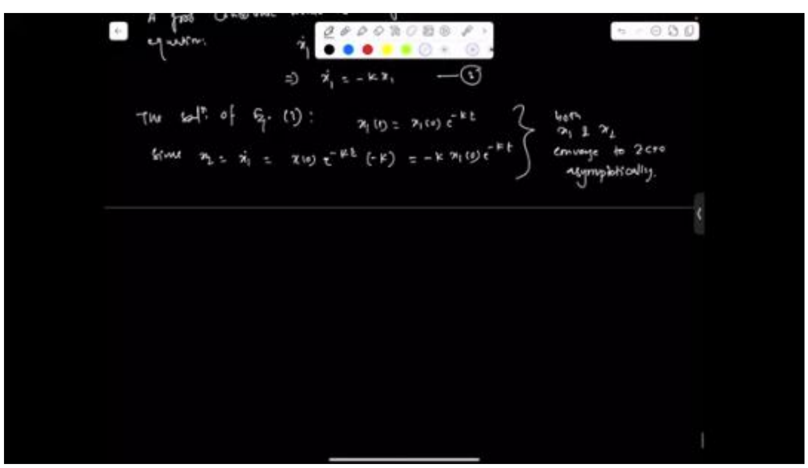

$$S = S(x_1, x_2) = x_2 + kx_1 \quad \dots Eq(4)$$

where k is greater than zero s, so here our motivation is or we can write to achieve this to achieve limit t tends to infinity x_1, x_2 equal to 0 with given convergence rate our convergence rate is, this is our convergence rate for x_1 , this, and x_2 , this, right, convergence rate, we can write here in bracket, e^{-kt}, in the presence of, bounded disturbance, if we have to let us define this equation for for we have to drive the variable is in equation four to zero infinite time by the control you know so we can say we can say that if s is goes to zero infinite time then also you can say x_1, x_2 also going to evolves over time in a finite time right so this can be achieved this part can be achieved if you have the Lyapunov function we have to choose a Lyapunov function right Lyapunov stability condition so now if you take time derivative of equation 4 we can write

$$\dot{S} = f + u + kx_2 \quad \dots Eq(5)$$

so our main motivation is we have to define the Lyapunov function in terms of s if the Lyapunov function is negative definite and if you can come of some structure what we'll be doing now then we can comment that x_1 and x_2 also will be stable if S is stable right so now we will define basically we are transforming the are transforming the variable we should define domain in s domain so now introducing the Lyapunov function Lyapunov function p in terms of is we can write

$$V = \frac{1}{2}S^2 \quad \dots Eq(*)$$

So we are defining the Lyapunov function in terms of S variables. So as for the asymptotic condition, we know that to provide asymptotic stability, stability of equation five about the equilibrium point, which is s equal to zero, because if you notice here, we have this is equal to zero, right? So here, we have denoted this equal to s. Also, the equilibrium point of this equation is zero. So we can write the following conditions must be satisfied. Basically, the first condition is $\dot{V}$ less than zero for s not equal to zero, and the second condition is limit $|S|$ goes to infinity and v also goes to infinity. This is what you already have done while studying Lyapunov's stability theorem. From the second condition,

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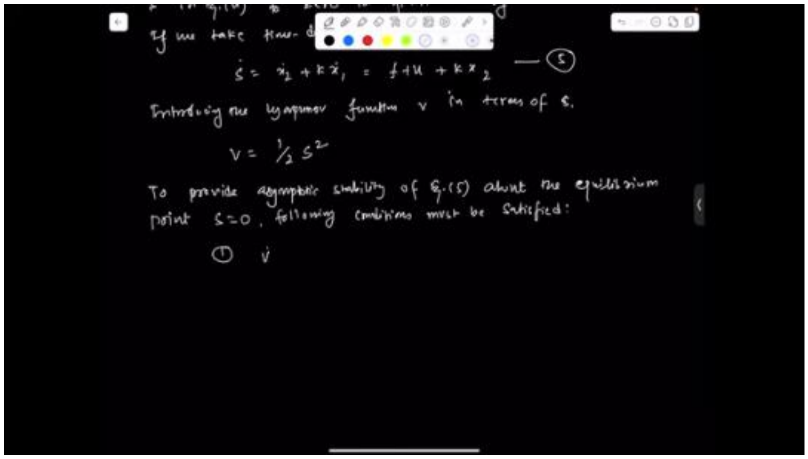

from the second condition, it is clear that t tends to infinity, which means t tends to infinity, right? And for this, the first condition is this, and the second condition is what you are getting. To achieve finite time convergence, condition one, which is this condition, can be modified as

$$\dot{V} \leq -\lambda V^{\frac{1}{2}} \;\ldots Eq(6)$$

where λ is greater than zero. So if you can write this $\dot{V}$ in this kind of structure, then we can say the system will be finite-time convergent. If you indicate this equation, Separating variables and integrating equation 6 over time from 0 to t, we get

$$V^{1/2} \leq -\frac{1}{2}\lambda t + V^{\frac{1}{2}}(0) \;\ldots . Eq(7)$$

If you integrate equation six and now, from seven, V(t) reaches zero in finite time t_r, which is bounded by this:

$$t_r \leq \frac{2V^{\frac{1}{2}}(0)}{\lambda} \;\ldots Eq(8)$$

So this is the expression for finite-time convergence of the states. This is a very important solution. Now, we'll prove the finite-time convergence of this control. We can write here: hence, the control input which is basically u, it will be designed now, is computed to satisfy equation 6, which is defined by this. So basically, we have to design so the main part is we have to design in this expression u here, u such that v dot will be in this form. So once you have this form, we can get the finite-time convergence expression like this tr. This is the main motivation for having this mathematical formulation. Now, satisfies 6, which will drive variables to zero in finite time, finite time, and stays there forever.

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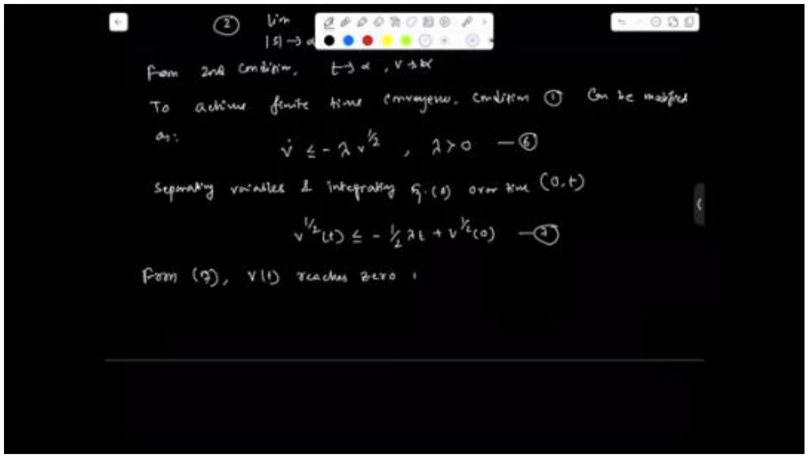

So now we'll apply this, which is our Lyapunov function—this expression, okay? So let us define the steps. Taking the time derivative, these are the parts; these are the steps to satisfy these conditions. Now we are going to move to our main problem, which is defined by star—the Lyapunov function, which is defined by S, and S is also a function of x_1 and x_2, right? Now, taking the time derivative of V in equation star, we have

$$\dot{V} = S\dot{S}$$

$$\dot{V} = S(f + u + kx_2) \quad \ldots Eq(9)$$

So now let's assume $u = -kx_2 + v$, where v is some signal function,

$$v = -\rho \, sgn\,(S), \quad \rho > 0$$

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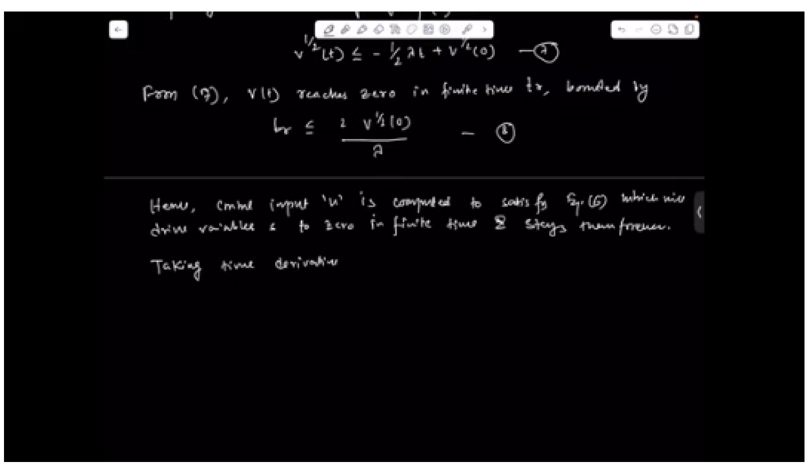

signum we can write as sgn(s), we can write

$$sgn\ (S) = \begin{cases} +1, & if\ S(t) > 0 \\ -1, & if\ S(t) < 0 \\ 0 & if\ S(t) = 0 \end{cases} \quad ...Eq(10)$$

So now we can write, if you substitute this u in equation 9, we can write

$$\dot{V} \leq |S|\Delta + Sv \quad ...Eq(11)$$

Now in this expression if you substitute the value of $v = -\rho\ sgn\ (S)$ in Eq 11.

$$\dot{V} \leq |S|(\Delta - \rho) \quad ...Eq(12)$$

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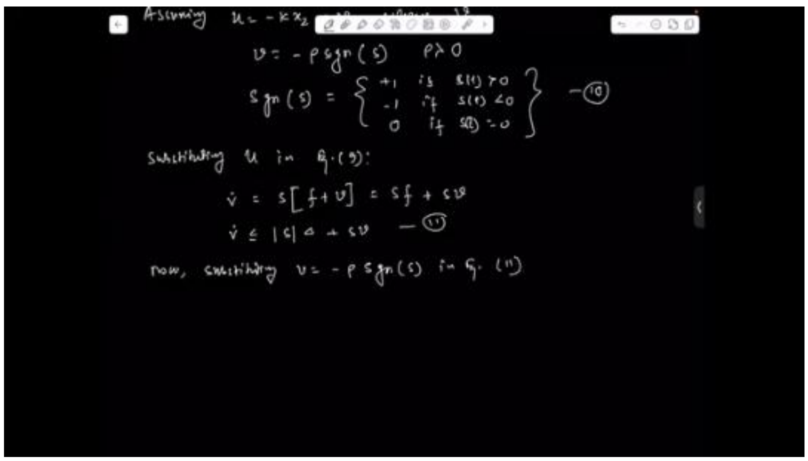

now also already we have this expression here equation 6 already we have this structure v dot equal you can write for the finite time convergence This is the expression we have for the standard sum for this standard conditions. So this is the one we have obtained for the finite term convergence for any S, right? You can write also we have from equation 6. from equation 6 we have

$$\dot{V} \leq \frac{-\lambda}{\sqrt{2}}|S| \quad ...Eq(13)$$

so now we can compare equation 12 and equation 13

$$\rho = \Delta + \frac{\lambda}{\sqrt{2}} \quad ...Eq(14)$$

so we have the row now so the control we can now substitute this v this thing this be our control algorithm you here we can substitute this being in control you here so you can substitute here like this so The overall control hence the total control is

$$u(t) = -kx_2 - \rho\, sgn(S) \;\; \dots Eq(15)$$

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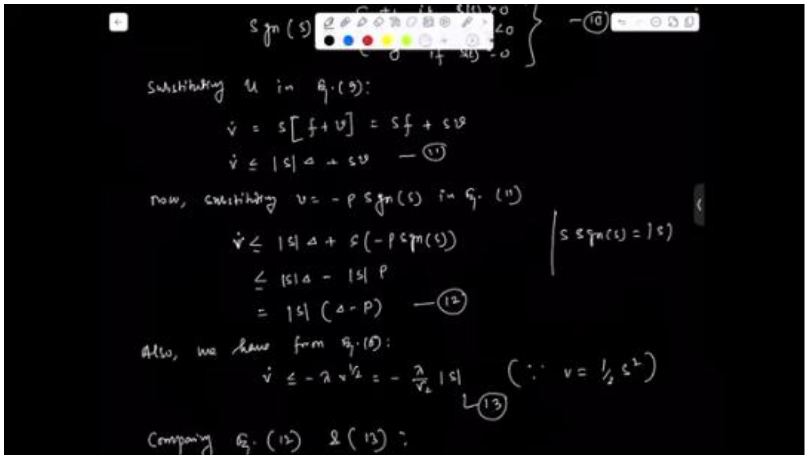

so this is how we can design the sliding mode control. But as you proceed slowly, you will understand more about this concept. While we'll be taking practical problems, we can come up with a better picture of how we can design sliding mode control. So now, in this u

$$u(t) = u_{eq}(t) + u_{sw}(t)$$

u_{eq} basically compensates the known dynamics, and $u_{sw}(t)$ ensures robustness against disturbance and finite time convergence. And this is called, this eq is called equivalent control, and this is called sliding control. So we can write from this equation:

$$u_{eq} = -kx_2$$

$$u_{sw} = -\rho\, sgn(S)$$

So this is our overall control sliding mode control for the dynamics one. We are actually understanding the problem, but as you proceed with other examples, it will be easier to design sliding mode control because we will be following some proper setup on how we can come up with the. Sliding mode control for the nonlinear system. In the next lecture, we'll be doing more observations. What are the different scenarios happening in sliding mode control? Because those are the parts required to understand more about sliding mode control.

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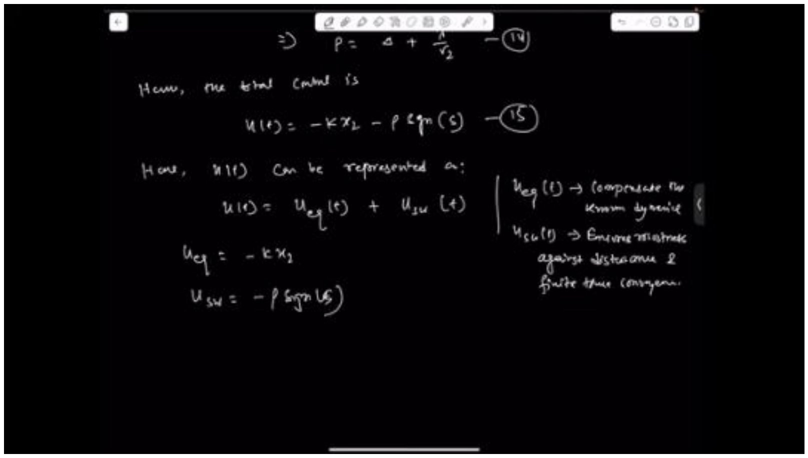

Because if you notice here, in the control, we have this function. And this is basically, if you notice, sgn of S, it will give minus 1 or plus 1 or 0. So due to this, in the control, we have some chattering. So how to tackle all these issues, we'll be discussing in detail in the next lecture. Thank you.