

Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 34

Krasovskii's Theorem

Hello everyone. In today's lecture, we will be discussing how we can find a suitable Lyapunov function that can provide the maximum region of stability. We also have an example to validate this concept. So, what you have understood from the last lecture is that if you have a positive definite function, and $\dot{V}(x)$, if it is negative definite or negative semi-definite, we can comment on the stability of the system without solving the dynamics. Now, the question is how we can find this positive definite function that can provide us the maximum region of stability. So, the question is how we can find the positive definite function that can provide the maximum region of stability. So here, we are going to consider the Krasovsky theorem. That is a very important theorem in nonlinear control synthesis.

So, here, through which we can find the Lyapunov function that can provide the maximum region of stability. So, the theorem is the Krasovsky theorem. So here, let me first state the concept, then we'll have the example. It provides a sufficient condition for asymptotic stability of an equilibrium point by constructing a Lyapunov function based on the system dynamics. So, for example, let us consider you have a system. Consider Consider a nonlinear system

$$\dot{x} = f(x) \dots Eq(1)$$

Where $x \in \mathbb{R}^n$, and $f(x)$ is a continuously differentiable function. Also, assume $f(0)=0$, meaning that the origin is an equilibrium point. It's an equilibrium point. Okay, so here, from this system, this is the system defined by equation number one. So, we can form a Jacobian matrix of the system. The Jacobian matrix we can form

$$F(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

The dimension of this matrix is n cross n because the system we have considered has dimension n. So now, let us define a function. Let us define function

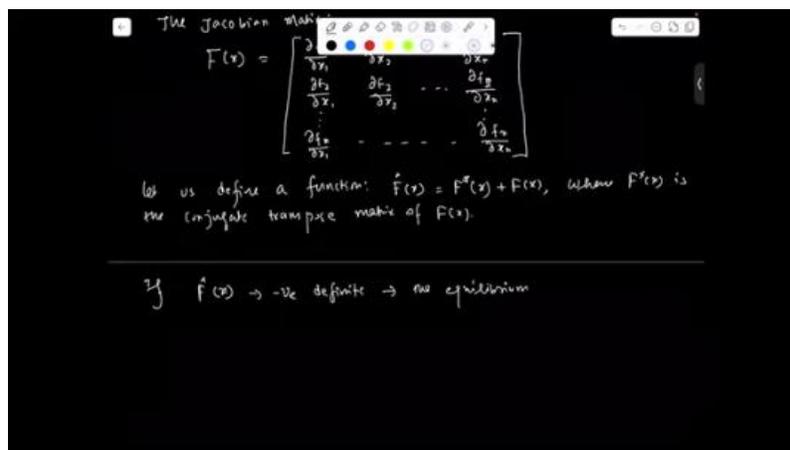
$$\hat{F}(x) = F^*(x) + F(x)$$

So here, $F^*(x)$ is the conjugate of the transpose matrix of $F(x)$. So now, according to the Krasovskii theorem, if $\hat{F}(x)$ is negative definite, negative definite, we can comment that the equilibrium point equilibrium point, which is x_e in this case, it is the origin, is asymptotically stable. Okay, so this is how we can come up with the stability condition for the system defined by equation one. This is one So now, we will formulate the derivations of this Lyapunov function for this equation for the system defined by equation one. So here, the Lyapunov function, Lyapunov function chosen as

$$V(x) = f^*(x)f(x)$$

So this is if $f(x)$ is a basically vector here, this is the vector. So here, this is basically it becomes the row vector if you take the conjugate transpose of the function $f(x)$. So here also, you're assuming $V(x)$, $V(x)$ goes to infinity if x goes to infinity.

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So, in this condition, we can say the equilibrium state, which is the origin, is asymptotically stable. In large, okay, so we are going to give the proof of this concept. Here, we have chosen $V(x) = f^*(x)f(x)$. One thing, this is the function, actually, this is the function. And this is a matrix. This f and this small $f(x)$, they are not equal. So now, if we take the time derivative of this $V(x)$,

$$\dot{V}(x) = (\dot{f}^*(x)f(x) + f^*(x) + \dot{f}(x)) \dots Eq(1)$$

$$\dot{f}(x) = F(x)f(x) \dots Eq(2)$$

Substituting equation two in equation one, we have

$$\dot{V}(x) = f^*(x)\hat{F}(x)f(x)$$

Now, if $\hat{F}(x)$ is negative definite matrix, then we can say $\dot{V}(x)$ will be negative definite. So we can write here if $\hat{F}(x)$ is negative definite, then $\dot{V}(x)$ will be negative definite. Now, this is how we can prove the theorem. Now, we will take an example to validate this concept.

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The image shows a handwritten derivation on a blackboard background. It starts with the definition of $V(x) = f^*(x)f(x)$. Then, it calculates the time derivative $\dot{V}(x) = (\dot{f}^*(x)f(x) + f^*(x)\dot{f}(x))$, labeled as equation (1). Next, it defines $\dot{f}(x) = \frac{\partial f}{\partial x} \frac{dx}{dt} = F(x)\dot{x} = F(x)f(x)$, labeled as equation (2). Then, it substitutes equation (2) into equation (1), resulting in $\dot{V}(x) = [f^*(x)F(x)f(x) + f^*(x)F(x)f(x)]$. This is simplified to $\dot{V}(x) = f^*(x)[F^*(x) + F(x)]f(x)$. Finally, it identifies the term in brackets as $\hat{F}(x)$, so $\dot{V}(x) = f^*(x)\hat{F}(x)f(x)$.

So, example. So, we have the nonlinear system

$$\dot{x}_1 = -x_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = x_1 - x_2 - x_2^3 = f_2(x_1, x_2)$$

So from this system, we can easily find the equilibrium point. So from this the equilibrium point, we can choose

$$f_1(x_1, x_2) = 0 \Rightarrow x_1 = 0$$

$$f_2(x_1, x_2) = 0 \Rightarrow x_2 = 0$$

Here, we can find or we can say this is one of the equilibrium points for the system, one of the equilibrium points, but there can be another equilibrium. So now here, we can form the function

$$f(x) = \begin{bmatrix} -x_1 \\ x_1 - x_2 - x_2^3 \end{bmatrix}$$

$$-x_2 - x_2^3 = 0 \Rightarrow x_2 = 0$$

$$x_2 = \pm i$$

Since we are considering the equilibrium point 0 0, that's why you have taken 0 and x=0. So now we can find the Jacobian matrix. So here we can form the Jacobian matrix

$$F(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -1 - 3x_2^2 \end{bmatrix}$$

Now we will find $\hat{f}(x)$ which is defined here. So we can use this function. So here

$$\hat{f}(x) = F^*(x) + F(x)$$

or we can write the transpose conjugate transpose of F(x)

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$$\hat{f}(x) = \begin{bmatrix} 2 & -1 \\ -1 & 2 + 6x_2^2 \end{bmatrix}$$

Now we will check whether this part is this matrix is positive definite or not. So we can verify you can use the Sylvester theorem and we can comment on the positive definiteness of $\hat{f}(x)$. So first we'll find q_1 what you have done before so it is true this is positive and

$$|Q_1| = 2 > 0$$

$$|Q_1| = \begin{vmatrix} 2 & -1 \\ -1 & 2 + 6x_2^2 \end{vmatrix} = 12x_2^2 + 3$$

so this is also positive definite because this is we have the square down here this is also positive so we can comment

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$$\begin{aligned} \hat{f}(x) = f''(x) + F(x) &= \begin{bmatrix} -1 & 1 \\ 0 & -1-3x_2^2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1-3x_2^2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ 1 & -2-6x_2^2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2+6x_2^2 \end{bmatrix} \end{aligned}$$

$F(x)$ is negative definite is negative definite so if it is the definite we can easily find this

$$V(x) = f^*(x)f(x) = x_1^2 + (x_1 - x_2 - x_2^3)^2$$

This is how we can find the Lyapunov function for the system. And I hope it is understandable how we can find the Lyapunov function for the nonlinear system. Now we will take another example to verify the concept. So here, example 2. We have the nonlinear system. Let us assume we have the nonlinear system.

$$\dot{x}_1 = -x_1 + x_2$$

$$\dot{x}_2 = -x_1 - x_2^3 \dots Eq(1)$$

and study the Lyapunov function or we can write find the Lyapunov function Lyapunov function of equation one so solution so here we can write we can form the function

$$f(x) = \begin{bmatrix} -x_1 + x_2 \\ -x_1 - x_2^3 \end{bmatrix}$$

And we can form the Jacobian matrix. So, I'm going through the steps and process. I'm not going through all the steps to find the solution. So, we can find the Jacobian matrix

$$F(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 - 3x_2^2 \end{bmatrix}$$

so here we'll find also the $V(x)$ as

$$V(x) = f^*(x)f(x)$$

this is our Lyapunov function to be designed. But before we comment on the Lyapunov function of this equation, first, we have to check $\hat{F}(x)$ whether it is coming to be negative definite or not.

$$\begin{aligned} \hat{F}(x) &= F^*(x) + F(x) \\ &= \begin{bmatrix} -2 & 0 \\ 0 & 6x_2^2 \end{bmatrix} \end{aligned}$$

So it can easily be commented that this matrix is positive definite, this matrix, and since there is a negative sign here, $\hat{F}(x)$ is negative definite. So we can write, say, we can comment, hence $f \cap x$ is negative semi-definite. So now we can form the Lyapunov function, $V(x)$.

$$V(x) = f^*(x)f(x)$$

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Find the Lyap

$$f(x) = \begin{bmatrix} -x_1 + x_2 \\ -x_1 - x_2^3 \end{bmatrix}$$

$$F(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -3x_2^2 \end{bmatrix}$$

$$V(x) = f^T(x) f$$

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$$f^T(x) f = \begin{bmatrix} -1 & 1 \\ -1 & -3x_2^2 \end{bmatrix} \begin{bmatrix} -x_1 + x_2 \\ -x_1 - x_2^3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -6x_2^3 \end{bmatrix} = - \begin{bmatrix} 2 & 0 \\ 0 & 6x_2^3 \end{bmatrix}$$

Hence, $\hat{F}(x)$ is negative semi-definite.

and if you multiply the row and column, we will have the Lyapunov function, which is

$$V(x) = (-x_1 + x_2)^2 + (-x_1 - x_2^3)^2$$

So this is the Lyapunov function for the given system. So, through this Lyapunov function, we can check the stability of the system of the equilibrium point of the system. So this is how we can form the Lyapunov function of any non-linear system. This is a standard procedure to find.

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Here, $\hat{f}(x)$

$$V(x) = -f^T(x) f(x) = \begin{bmatrix} -x_1 + x_2 & -x_1 - x_2 \end{bmatrix} \begin{bmatrix} -x_1 + x_2 \\ -x_1 - x_2 \end{bmatrix}$$
$$= (-x_1 + x_2)^2 + (-x_1 - x_2)^2$$

→ Lyapunov function

Here we are stopping the Lyapunov stability analysis, and from the next lecture onwards, we'll be doing the control part. Thank you.