Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 33

Lyapunov Stability Theorem

Hello everyone. In today's lecture, we will be discussing the stability theorem, the Lyapunov stability theorem. We will also have examples and the MATLAB simulation for an example. Here, first we will state the Lyapunov stability theorem. Let us consider a system described by, for a system described by $\dot{x} = f(x,t)$ and this function f satisfies for 0 states that means where $f(0,t) = f(0)$ for all t greater than or equal to t_0 . Now if There exists a scalar function vxt, which has continuous partial derivatives partial derivatives and satisfies the following conditions. The first condition is $V(x,t)$ is positive definite and the second condition $\dot{V}(x,t)$ is negative definite then Then the equilibrium point, let us say xe, is uniformly asymptotically stable. So, for a particular system, we will consider a positive definite function and if the time derivative of that positive definite function is decreasing over time, then we can say the equilibrium point, which is basically the origin for the system, will be uniformly asymptotically stable. So, now let's take an example and validate this theorem example. Study the stability of the system given by

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$

So, this is basically a natural system, and we will study the stability of this natural system and based on that, we can conclude whether the system should be controlled or not. So, in the last few lectures and in this lecture also, we are talking about the stability of the natural system. Now, we will find the equilibrium point for the system. The equilibrium point for this particular system will be $x_e = (0,0)$. You can easily find it. We have done a lot of examples in our previous lectures on how to find the equilibrium point. Now, here we will find a positive definite scalar function. So, here we will choose Let us choose a positive definite scalar function $V(x) = x_1^2 + x_2^2$

So, it is basically randomly chosen, but if you choose any other function, this condition should satisfy. We have chosen this function, and this is also a positive definite function. Now, we will take the time derivative of this function and validate the second condition. So, here taking the time derivative

$$\frac{\partial V}{\partial t} = -2(x_1^2 + x_2^2)^2 \ldots Eq(1)$$

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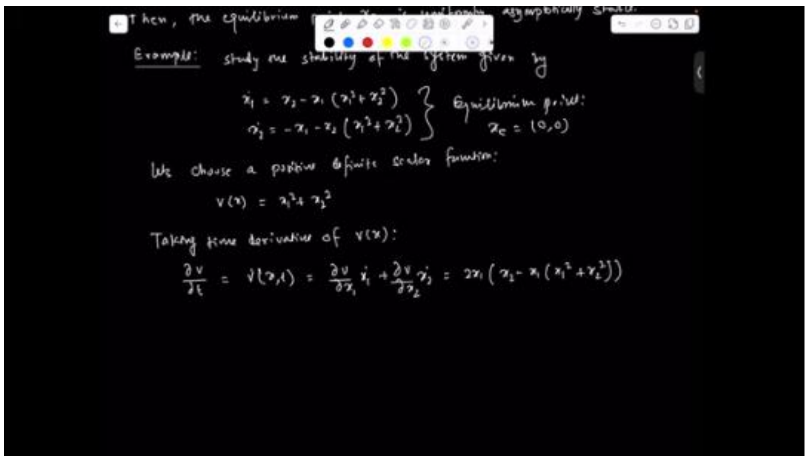

So, from this expression, it is clear that $\dot{V}(x,t)$ is negative definite. Now, we'll check how the system is going to behave at the equilibrium point. So, at $x_1 = x_2 = 0 \Rightarrow \dot{V}(0,t) = 0$ So, we can say at the equilibrium point at 0, 0, it is 0, and at other points, it is negative. Negative at all other points. So, our stability theorem is satisfied for this particular example. For this particular example, it is satisfied. So, we can write according to Lyapunov's theorem, the equilibrium point which is basically (0,0), is asymptotically stable. And also,

$$\|x\| \to \infty \Rightarrow \|V(x,t)\| \to \infty$$

So, equation 1 is asymptotically stable in the large because you are taking the large values of x_1, x_2. So, this is how we can study the stability of the system without solving the equation. This is a very, very powerful tool to analyze the system. Now, we will take another example, and we will find the MATLAB simulation for that particular system and also comment on the stability of the system.

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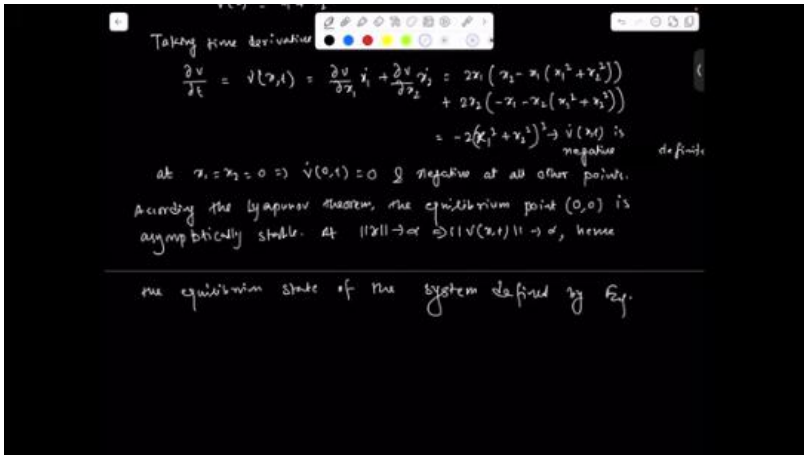

Example 2: Consider a system described by

$$\dot{x}_1 = -x_1 - x_2 e^{-t}$$

$$\dot{x}_2 = x_1 - x_2 \ldots Eq(2)$$

Study the stability of the system using Lyapunov's theorem. So here, similar to the above example, first we have to find the equilibrium point of the system solution. The equilibrium point, the equilibrium point of equation 2 yields to be $x_1 = x_2 = 0$. Now let us select a So we have the equilibrium point now. Now we'll study the stability of the system at the equilibrium point. For that, first we have to choose the Lyapunov function.So let us select, let us choose a positive definite function, which is nothing but the Lyapunov function.

$$V(x,t) = x_1^2 + x_2^2(1 + e^{-t})$$

This is the function we have chosen, but if you want to choose, you can choose yourself, but the condition needs to be satisfied: it should be positive definite. Now from this function, we can easily see that $V(x,t) > 0$ for all $x \neq 0$. Now we'll find the derivative of this function. For that, we can use this

$$\frac{\partial V}{\partial t} = -x_2^2 e^{-t} + 2x_1\dot{x}_1 + 2x_2\dot{x}_2(1 + e^{-t}) \ldots Eq(3)$$

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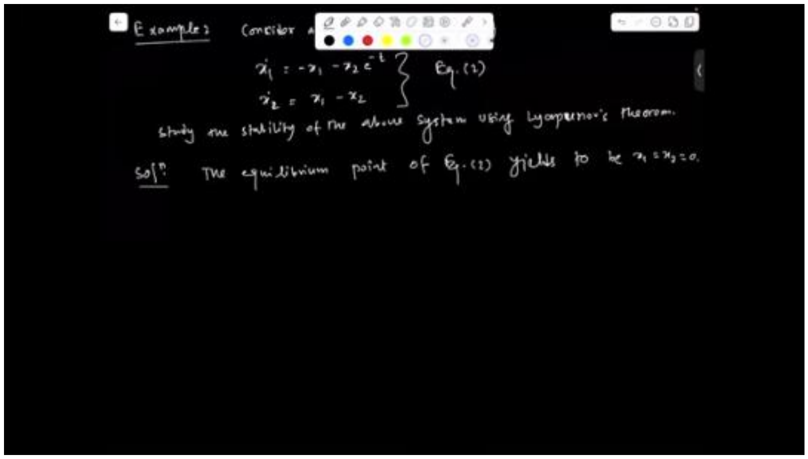

Now, substituting equation 2 in equation 3, we have

$$\frac{\partial V}{\partial t} = -2(x_1^2 + x_2^2 - x_1 x_2) - 3x_2^2 e^{-t}$$

So here, as t tends to infinity, we can write e^{-t} goes to zero, which is obvious. In this condition, we can write dv dt to show how the energy is going to decay over time. So, if you take the time tends to infinity, we can write

$$\frac{\partial V}{\partial t} = -x_1^2 - x_2^2 - (x_1 - x_2)^2$$

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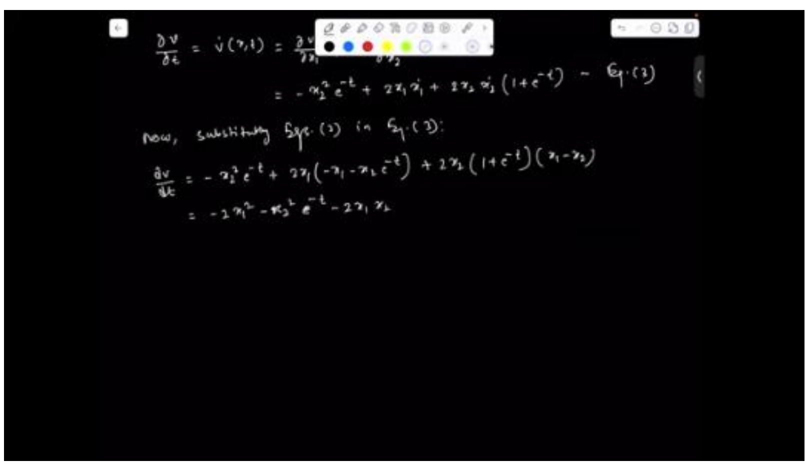

So, the whole value of $\frac{\partial V}{\partial t}$ is negative. So now we can write. So, hence we can write dv/dt is less than 0 for all $x_1, x_2 \neq 0$. So, from this we can write, hence the equilibrium point Or you can write simply, the system is stable, basically.

You can write the system is stable. And here, further, if you do more simply, because you can write on V(x), you can write also further. We can write the norm of X,

$$\|x\| = \sqrt{x_1^2 + x_2^2}$$

which is also infinity as t tends to infinity. So, which implies that

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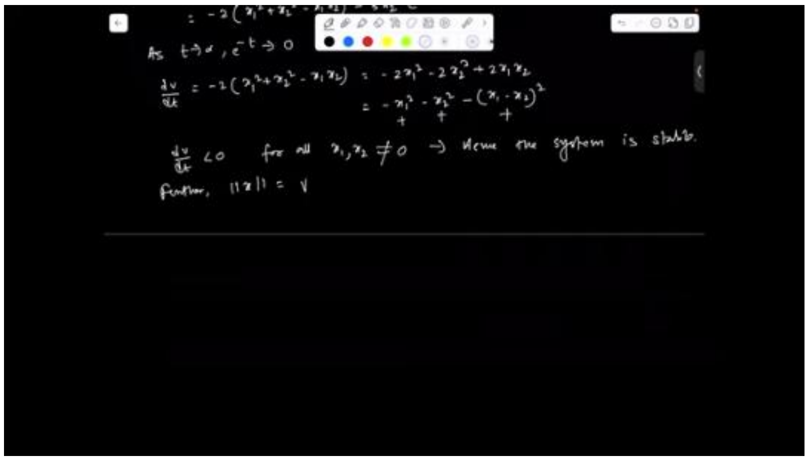

$$V(x,t) = x_1^2 + x_2^2(1 + e^{-t})$$

where we can write $x_1, x_2 \rightarrow \infty$ if that so therefore all sufficient conditions of this function are validated so hence we can write hence we can write the origin which is $x_1 = x_2 =$, which is the equilibrium point also, is asymptotically stable equilibrium point. Equilibrium point. So, this is basically the same way what we have done for example 1. What we have done here, the same properties also we have followed for example 2. Now, we'll see the MATLAB code for this example. And we'll see whether the system goes to the equilibrium point, which is (0, 0), or not. So, we have the code for the system. So, this is the MATLAB code for this example. So, you can notice that we have chosen two states.

We are simulating; we have taken the initial condition for x_1 as 10 and x_2 as 5. And we are simulating this code for 10 seconds. And we have used ODE45 to find the solution of these ODEs. This is the function we have defined. And these are the variables to be plotted.

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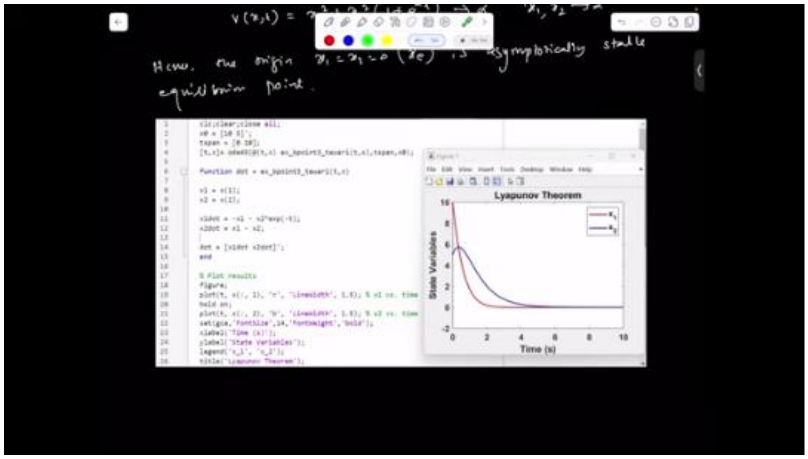

These are our two first-order ODEs. And this is basically the plot, showing how to see the labels in XY. This is basically the MATLAB code. And we can see that x_1 and x_2 , starting from this initial point, over time, are going to zero. The system is stable. So, this equilibrium point, which is (0, 0) in this example, over time, the system response goes to zero. So, this system is absolutely stable. And this is also validated through the analysis we have done in this $\dot{V}$. It is also found that the system is stable; basically, the equilibrium point is stable.

And also, you can see through this response as well. So, this is how we can analyze the system without solving the equations of the system's dynamics. In the next lecture, we'll be discussing how we can find the Lyapunov function, which is a very important part. After that, we'll proceed to the control design process. Thank you.