Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 32

Sign Definiteness of Scalar Functions

Hello everyone. In today's lecture, we will be discussing the basics of studying the Lyapunov second method. And in this part, we will be discussing the sign definiteness of the scalar function and Sylvester's theorem. So, why the scalar function? Because the energy we are considering for the system is basically a scalar function and also positive.

Why do we need to study this topic? Because most of the time, it is difficult to find the scalar function or the energy function for the system. So, this topic is very important to find the energy of the system. And also, in Sylvester's theorem, we will be coming up with the concept which will help us to find the positive definite function. Basically, the energy function for the system is scalar and also positive.So, how can we come up with that positive definite function? Sylvester's theorem can help us. So, the first part is sign-definiteness. So here, we'll come up with the positive definiteness. And let's assume we have the function

$$V(x) = V(x_1, x_2, \ldots, x_n) = \begin{cases} +ve, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

So, if the function is defined for n states and if it is positive definite, then v of x will be Positive for any in the state space for any value of x, so we can say for any value of x is not equal to 0, and if it is 0, if x is equal to 0, if the function satisfies this condition, we can say the function is positive definite. And the second part, positive semi-definite. Semi-definite means.

$$V(x) = \begin{cases} +ve, & \rightarrow \\ 0, x_n, x = 0 \in x_n \end{cases}$$

So, here the $V(x)$, this $V(x)$ will be positive in the state space for any value of x except one or more points in the state space, including the origin x equal to 0, where $V(x)$ will be 0. So, we can write $V(x)$ will be It will be positive for any state in the state space except some point or more points and x equal to 0, which belongs to xn.

This $V(x)$ will be 0. So, it means you can write here. Let me write here. V is said to be positive at all points in the state space except one or more points in the state space, including the origin x equal to zero, so where $V(x)$ will be zero. So, once the function follows this definition, we can say the function is positive semi-definiteness. Now, the third part is negative definiteness. So, here the function $V(x)$ is a scalar function, and it will be negative for all x all x not equal to zero and zero at x equal to zero. If the function follows this rule, then you can see the function is negative definiteness. And the third is negative semi-definiteness.

$$V(x) = \begin{cases} -ve, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

So, here actually, we can come up with the same definition what you have proposed here. The $V(x)$ will be, and the $V(x)$ is said to be positive. Instead of positive here, we will say negative at all points in the state space except one or more points in the state space, including x equal to 0, $V(x)$ equal to 0. So, the same definition except this, instead of positive, we will write negative. So, I'm not going to do that. So now, this is how we can define the negative definiteness and positive definiteness. Now, we'll come up with some, we'll consider a quadratic function, which can define the total energy of the system.

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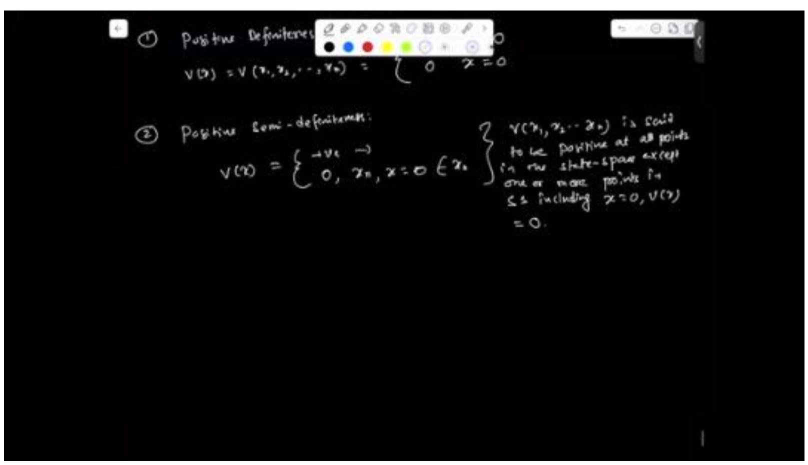

So, let me try it. Basically, in most of the systems, what you see practically now in daily life, the dynamical system, the total energy can be defined by the quadratic formula, so we can write the total energy for a dynamical system can be defined in a quadratic form as

$$V(x) = \sum_{i=1}^{N} \sum_{j=1}^{N} K_{ij} x_i x_j$$

So, we can $x_i x_j$ where here I'll say we can say K_{ij} are constant some constant parameter so as I have to say that most commonly used quadratic function in matrix form we can write

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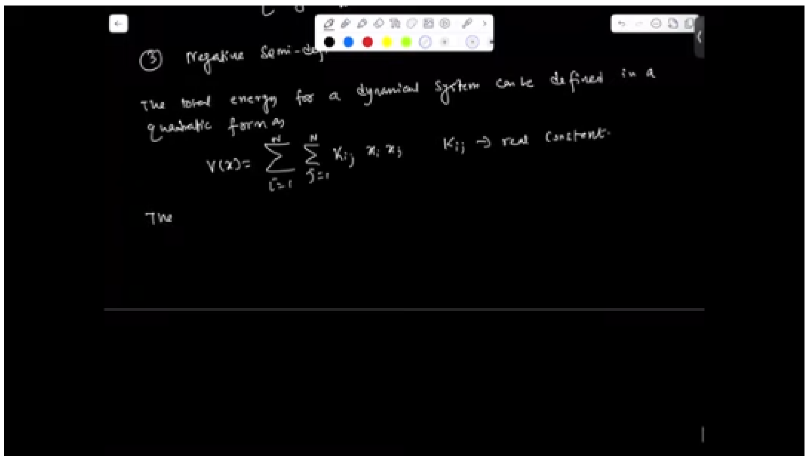

$$V(x) = X^T Q X$$

$$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & \cdots & \cdots & q_{nn} \end{bmatrix}$$

$$X^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

$$V(x) = q_{11}x_1^2 + q_{22}x_2^2 + \cdots + q_{nn}x_n^2 + \cdots (q_{ij} + q_{ji})x_i x_j + \cdots \ldots Eq(1)$$

So now, if you take a simple example, for example, we have the function

$$V(x) = x_1^2 + x_2^2$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so we can write It is a scalar function and also positive, so we can consider this the energy of some system, which can also be defined in this form where this matrix is a positive definite matrix. So the most important thing is how to find this matrix and based on which you can find the total energy of the system. Now, from this, we can write From this,

$$q_{ij} = \begin{cases} k_{ij}, & i = j \\ \frac{1}{2}\left(k_{ij} + k_{ji}\right) = q_{ji}, & i \neq j \end{cases}$$

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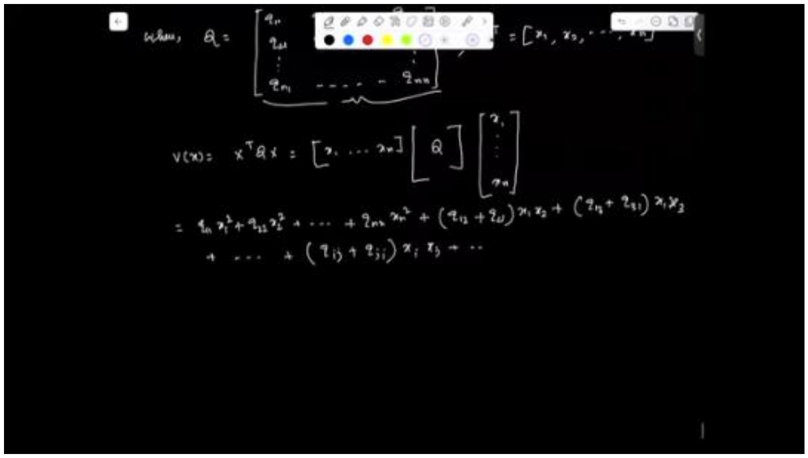

Let V(x) be some function

$$V(x) = 4x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + x_2x_3 + 2x_1x_3 \ldots Eq(2)$$

Now, if you compare both these equations 1 and 2,

$$k_{11} = 4, k_{22} = 2, k_{33} = 1, k_{23} = 1, k_{13} = 2$$

Now, if we compare this case with q_1, q_2 like that, so we can write here

$$q_{11} = k_{11} = 4$$

$$q_{22} = k_{22} = 2$$

$$q_{33} = k_{33} = 1$$

$$q_{12} = \frac{1}{2}k_{12} = 1$$

$$q_{23} = \frac{1}{2}k_{23} = \frac{1}{2}$$

$$q_{13} = 1 = q_{31}$$

$$Q = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 1/2 \\ 1 & 1/2 & 1 \end{bmatrix}$$

$$V(X) = X^T Q X$$

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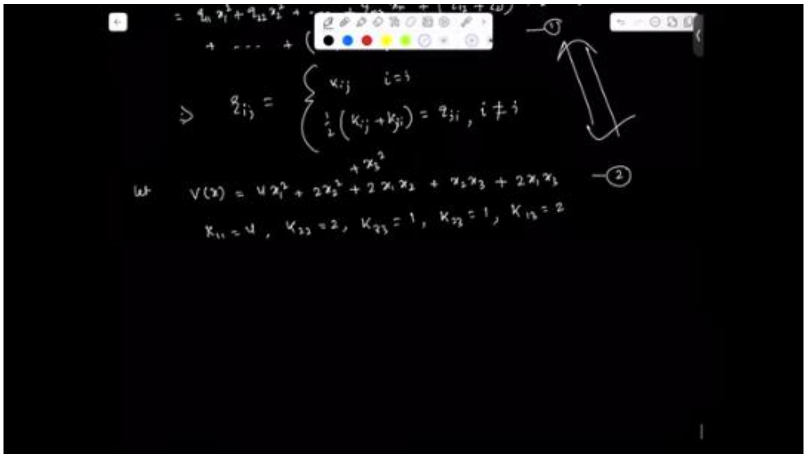

Now, if you use this for if you use this matrix and if you multiply this matrix with x transpose pre-multiply and x with post-multiply, we can get this Lyapunov function. So this is how we can find the Lyapunov function, but the most important part is how to find this. This positive matrix, so here for the definiteness, so now we are doing the second part of this in this lecture: positive definite definiteness. So a quadratic function, a quadratic function.

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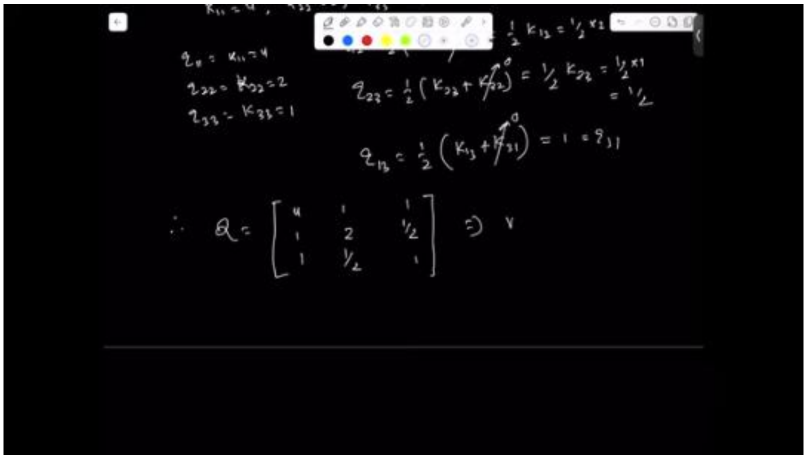

$V(X) = X^T Q X$ is positive definite if and only if only if all the principal. Principal determinants, which is we can define $|Q_1, Q_2 \ldots Q_n|$, but after this, we will explain now just let me wind up this definition of the matrix Q are positive. Here, if you let's let us assume we have the matrix Q, and if it is defined by

$$Q = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|Q_1| = |a|, \quad |Q_2| = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

$$|Q_3| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

so this is basically the third principal determinants. Now, um, we'll look something sign definiteness. Okay, it is actually you can do can just simply you can go the example. I think it is better instead of this say example. Let us consider we have the some function

$$V(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 1 & 2 & 1/2 \\ 1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

And we have the matrix for the same matrix what you have done what you have obtained here, the same matrix I'm assuming here, and we will check the positive definiteness for this particular case using this Sylvester theorem.

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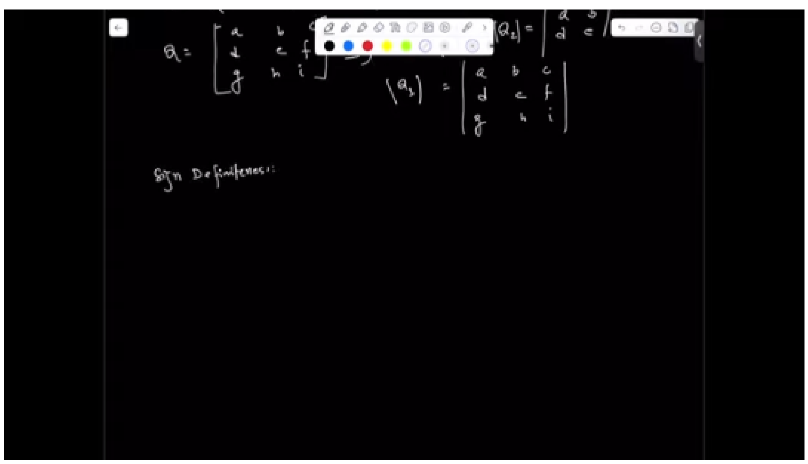

$$|Q_1| = 4, \quad |Q_2| = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 7 > 0$$

$$|Q_3| = \begin{vmatrix} 4 & 1 & 2 \\ 1 & 2 & 1/2 \\ 1 & 1/2 & 1 \end{vmatrix} = 5 > 0$$

So we can say that all the principal elements are positive, and we can say this is a positive definite function. So this is how we can find the V(x) function, and also, if you need to find the matrix inside the skew matrix, we can also follow the same procedure as we have done before. So this is very important for understanding the second theorem, and we will be using this kind of function in our example, which we will be taking later. Also, it is very important that the structure of the Lyapunov function for an nth order system Sometimes it is difficult to choose the Lyapunov function, but if you know how to find the positive definite, we can simply find it. Anyway, we'll have a lot of examples in the later lecture, and also how to use, how to find the Lyapunov function for any other dynamical system, we'll go through in detail. So let's stop it here, and we'll continue from the next lecture.

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