Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 31

Lyapunov's First Method

Hello everyone. In today's lecture, we will be discussing how we can do the stability analysis of the system without solving. Here, we are going to study the Lyapunov method, and based on which you can comment whether the system is stable or not. And also, we will cover Lyapunov's first method, where we can study the system stability by solving the full dynamics of the system. Before we proceed to the first method, we will come up with a motivational example of how we can do the analysis for the nonlinear system.

So, let us have an example. Here, we are going to consider a very simple system: the mass-spring-damper system. The mass is attached to our wall via the spring. This is the spring, and this is the damper. The position of the mass, let's define x from the equilibrium point. The simple force balance equation for the system we can write is And this displacement x is there because of the force being applied on the mass. So, the force balance equation we can write is

$$F = m\ddot{x} + b\dot{x} + Kx \ldots Eq(1)$$

where you can assume a is the acceleration which is nothing but the second derivative of x with respect to t, and x is the displacement of the mass, and $\dot{x}$ is the velocity, which is nothing but dx/dt. And here, we are going to, since we are going to study the system and what you have done in the previous lectures is that, for studying the nonlinear system, we convert the system into autonomous form. For the autonomous system, as we have done before, the force is assumed to be zero.

$$m\ddot{x} + b\dot{x} + Kx = 0 \ldots Eq(2)$$

And here, let us assume the mass of, this is the mass.

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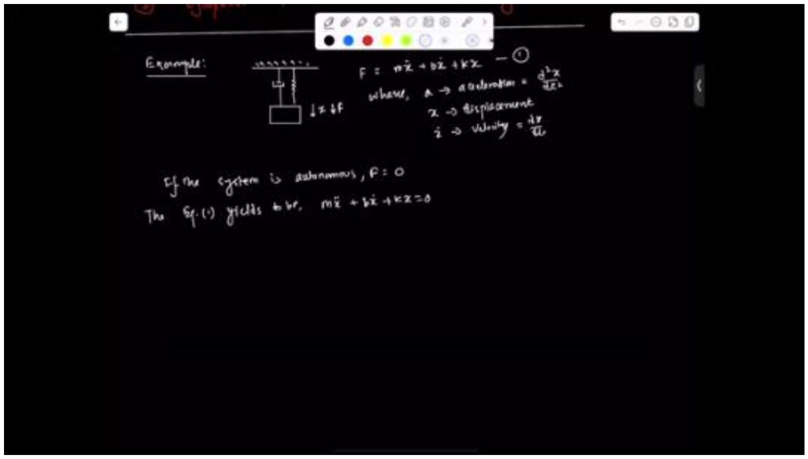

So, let us assume the mass m equal to 1. For m equal to 1,

$$\ddot{x} + b\dot{x} + Kx = 0 \ldots Eq(3)$$

Now, if you notice, the system is basically a second-order system. The system is a second-order system, and here we will convert the system into first-order ODEs. So, for that, we have to apply the change of variable. So, here let us define or we can write the system in state-space form. We can write

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -bx_2 - kx_1$$

The change of variable basically uses $x_1 = x$ and $\dot{x} = \dot{x}_1 = x_2$, So, using this variable, we have come up with the system into first-order ODEs. Now, since in the Lyapunov method, we will be using some energy function, through which we can check the system stability, it is different from the earlier lecture where we used the phase portrait method. So, here let us define the total energy of the system at any time. So, the energy, we can write the total potential and kinetic energy. We can write the total energy content in the system

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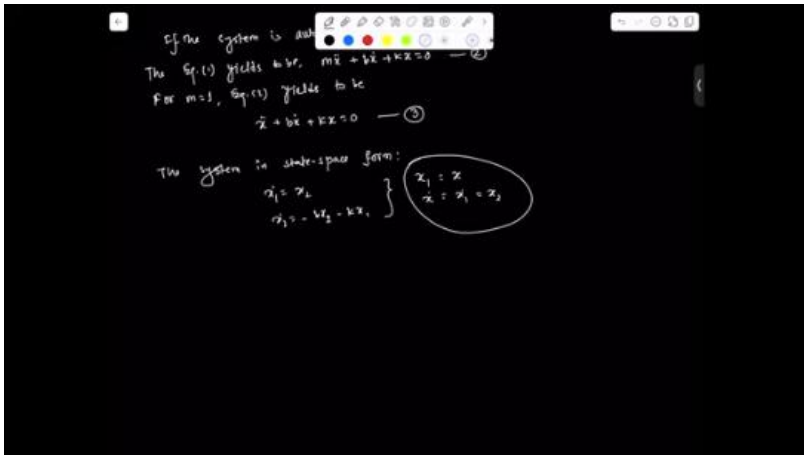

It is defined by V(x_1, x_2). So, V is the total energy in the system. Now we are defining it by

$$V(x_1, x_2) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}v^2 + \frac{1}{2}kx^2$$

Now, we will see how the time derivative of this V is going to behave. We will have a lot of stuff in later lectures. Why we need to consider V dot to be a decreasing function or increasing, all this stuff we will discuss in more detail, and how they are connected to stability. So here, basically, we have to check how the system's total energy is going to behave. So here, we will take the here we can write. This is the condition we have to follow.

$$V(x_1, x_2) > 0 \quad for \;\; x_1, x_2 \neq 0$$

so this the condition should be valid for this energy function, and

$$V(x_1, x_2) = 0 \quad for \;\; x_1, x_2 = 0$$

So, if this condition is satisfied, we can say this is the energy function for this particular system. We will have details in the later lecture on how this condition being we can write here. So, now we will see how the total energy of the system is going to behave. Let's see if the energy of the system is increasing or decreasing with time. So, the rate of energy, which is equation 4, or we can define this as equation number 5, and this is equation 4. And the rate of energy of equation we can write

$$\frac{dv}{dt} = \frac{\partial v}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial v}{\partial x_2}\frac{dx_2}{dt}$$

$$\frac{\partial v}{\partial x_1} = kx_1 \qquad \frac{\partial v}{\partial x_2} = x_2$$

$$\frac{dv}{dt} = kx_1x_2 + x_2(-bx_2 - kx_1) = -bx_2^2$$

So now if you notice here, this is basically this part is always positive, this part is always positive, and this is negative. The whole term is negative.

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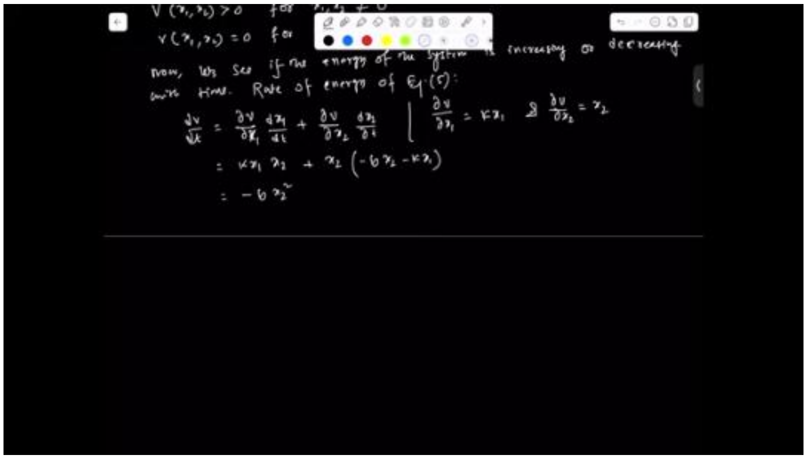

So it means, $\frac{dv}{dt}$ is actually decreasing over time. And we can write here, and it is decreasing when the system goes to the equilibrium condition. And at equilibrium, the total energy of the system will be zero. So we can write here, the energy of the system is decreasing until the equilibrium condition equals zero.

So here, why is this equilibrium condition zero? Because from this equation, from this equation, we can write the equilibrium condition. We can write from this expression. Equilibrium condition $\dot{x}_1 = 0 \rightarrow x_2 = 0$ And from the second equation, $\dot{x}_2 = 0 \rightarrow x_1 = 0$. The equilibrium condition, equilibrium condition, we can write this as

$$x_e = [x_1 \quad x_2]^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, at the equilibrium point with the function, sorry, the energy of the system, which you know, we can write $\frac{dv}{dt} = 0$ at the equilibrium point and for this v equals constant. So, this is how we can comment on the system, the system, the autonomous system, autonomous system is stable. And why it is autonomous, we are considering because we have no function, we have no it context later, so this is how we can check the stability of the system without solving the dynamics or equation, and this is a very, very powerful tool to analyze the system. It should be noted that the total energy of the system sometimes is difficult to find, but we have some generalized tools on how we can find the total energy of a system later in this context. And now we'll move to the, so I hope this part is clear, the first part, how we can check the stability of the system without solving the whole equation of motion. Later, we'll have more examples of how we can do the same stability analysis for any other dynamical system. And this is just a motivational example. Now, we will move to the second part in this lecture. We have our first method to study the nonlinear system. Here, we will go through some basics. Already, we have done this part before. How can we linearize the system? So, here, let us assume we have a nonlinear system. Let us consider. Let us consider time-varying, time-invariant, time-invariant nonlinear system

$$\dot{x} = f(x, u)$$

where $f(x, u)$is continuously differentiable. So here, we are assuming x is the state and u is the control or any other perturbation in the system. So here, we can assume this is the nonlinear system. And now we're going to see how we're going to look, how we're going to find a linear model of the nonlinear system. This part already we have done multiple times in our previous lectures, but again we'll do this part because it is very important to study the Lyapunov first method. Here, let's assume the equilibrium point of the system.

Let us consider the equilibrium point. The equilibrium point is defined as $x_e, u_e.$ This equilibrium point is found for the system, and at the equilibrium point, the function will satisfy. We can write at the equilibrium point, at x_e, u_e, we can write

$$f(x_e, u_e) = 0$$

and here, actually, we can write the control u can be some 0 or constant, some value 0 or k, we can write. So, here 0 or constant, some constant value we are applying if there is control, or unless it is 0, it is an unforced system. In the presence of perturbation, let us assume in the presence of perturbation, the system will deviate from the equilibrium point, as we have learned in our previous lectures, that we can write

$$x(t) = x_e + \delta x(t) \dots Eq(1)$$

$$u(t) = u_e + \delta u(t) \dots Eq(2)$$

So, here $\delta x(t)$ and $\delta u(t)$ are perturbations in x(t) and u(t). Now, we will take the time derivative of the equation. And if you take the time derivative of equation 1, taking the time derivative, of equation 1, we can write

$$\dot{x}(t) = \dot{x}_e(t) + \delta\dot{x}(t) \rightarrow \dot{x}(t) = \frac{\partial}{\partial t}(\delta x(t))$$

So, it is a function of time we can write. and since the equilibrium point is assumed to be a constant parameter or, sorry, constant value, we can write

$$\dot{x}(t) = \dot{x}_e(t) + \delta\dot{x}(t) - f(x_0 + \delta x, u_0 + \delta u(t))$$

this expression in our non-linear system here and if you notice this function, this is this function will apply here the Taylor series expansion, so using Taylor series,

$$\dot{x}(t) = \delta\dot{x}(t) = f(x_e, u_0) + \frac{\partial f(x_e, u_e)}{\partial x}\delta x + \frac{\partial f(x_e, u_e)}{\partial u}\delta u +$$

$$\frac{1}{2}\frac{\partial^2 f(x_e, u_e)}{\partial x^2}(\delta x)^2 + \frac{1}{2}\frac{\partial^2 f(x_e, u_e)}{\partial u^2}(\delta u)^2 + h(x_e, u_e, \delta x, \delta u)$$

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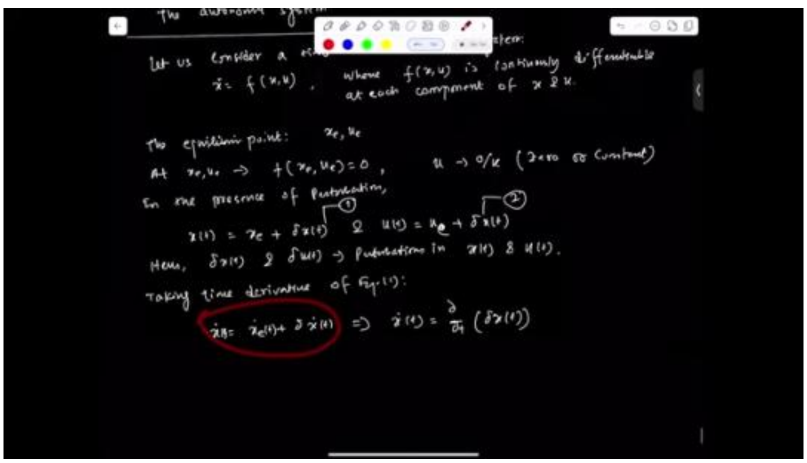

h is the the higher-order terms. So, the rest of the terms we are not writing. So, we can write these as higher-order terms. This you can write. So, now what will we do?

If Δx is very small, if you assume that Δx is very small, then these whole terms we can ignore in the equation because these terms are assumed to be very small perturbations from the equilibrium point, as here we are studying the local stability of the system. So, we can ignore these higher-order equation terms. We can write this expression, we can write.

$$\approx f(x_e, u_e) + \frac{\partial f(x_e, u_e)}{\partial x}\delta x + h(x_e, u_e, \delta x, \delta u)$$

So here we can write ∂f/∂x, actually the Jacobian matrix for the state, so we can write this expression:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

so this is what we call the Jacobian matrix and we can find this value maybe the constant value at some attics we can add x_e, u_0 if you substitute these values we can come up with some matrix and similarly for the control we can write

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}$$

this is what we can call the control part here we are not assuming the system is autonomous or not just we are taking the general system x u now we can do one thing if we assume if you assume $\delta x, \delta u$ actually the small values this goes to zero the higher order terms go to zero at a first order at a first order rate than the first order terms or we can write

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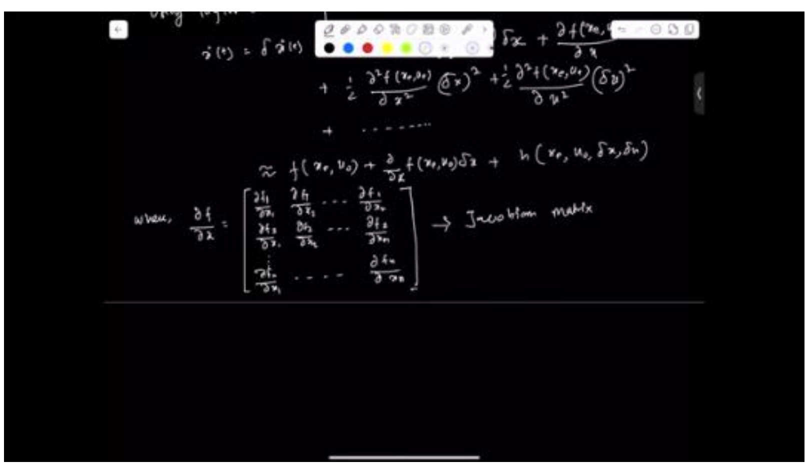

$$\lim_{\substack{\|\delta x\|\to 0 \\ \|\delta u\|\to 0}} \frac{h(\delta x, \delta u)}{\|\delta x\|} = 0, \qquad \lim_{\substack{\|\delta x\|\to 0 \\ \|\delta u\|\to 0}} \frac{h(\delta x, \delta u)}{\|\delta u\|} = 0,$$

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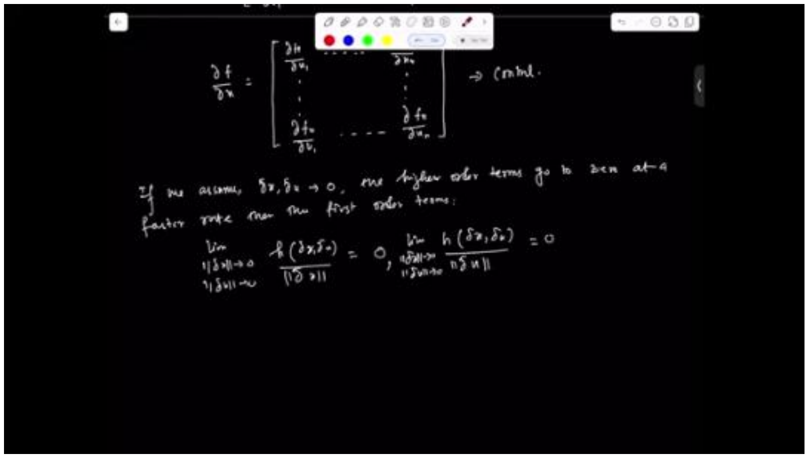

So let me write one word in a very small region of x and very small change in control input the behavior of the part of system can be approximated by the locally linearized system or linearized equations or we can write delta x dot b So here we can write so here we can write this expression equal to this expression sorry this expression and this expression anyway this expression here this expression is going to be 0 because at equilibrium point this functional will satisfy so here we can write

$$\delta\dot{x}(t) = \frac{\partial}{\partial x} f(x_e, u_0)\delta x + \frac{\partial}{\partial u} f(x_e, u_0)\delta u \ldots Eq(3)$$

You can say equation three is basically a linear and time-invariant equation. So, this is how we can convert the nonlinear system if you have a nonlinear system. And we can, if we apply this Taylor series expansion, we can get the LTI system. And this part has already been done multiple times. If you go back to our previous lectures, we have done it.

But why are you doing it now? Because we need this part to method. Now, let us take an example, and we will get a better picture of this. So, here, as an example, we can do one thing: we can take this example maybe later, or maybe we can directly connect how this concept is going to connect to the our Lyapunov first method, as we have discussed, that the Lyapunov method is actually valid for the autonomous system. So, we can write that the Lyapunov first method only applies to the autonomous system. This is a very, very important note. autonomous system.

So, we can write $\dot{x} = f(x)$. So, here, we are not going to consider the control in the system, only how we can, if the system is nonlinear, and how we can get the linear form. This has already been done. So, for this case, we will not have this term, this Jacobian matrix; we will have only this part. And also, u naught will not be there. So, it is easy to get the linear system for this autonomous system. So, now, for an autonomous system, we can write, for an autonomous system,

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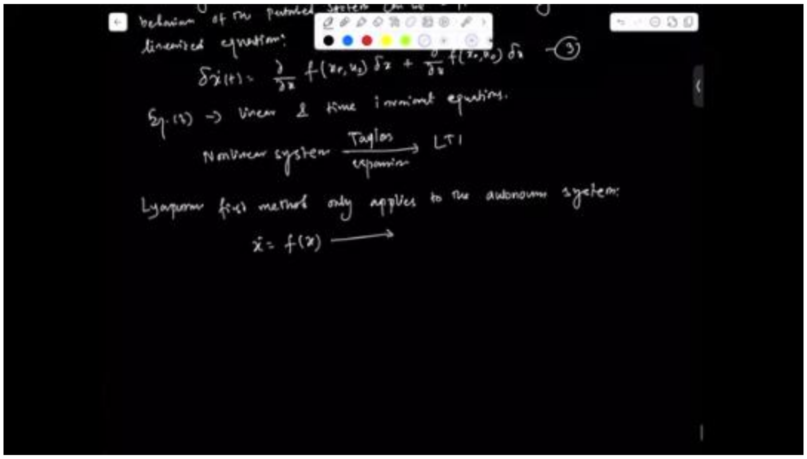

We can get the part of the system, part of the system

$$\delta\dot{x} = \frac{\partial f(x_e)}{\partial x}\delta x + h(x_e, \delta x) \ldots Eq(4)$$

So, this is basically easier if you consider an autonomous system. We'll have this expression instead of this. Instead of this, we will have that expression, right? It is basically u_0 will not be there. So, the derivations will be easier for the autonomous system. And similar to the previous, we can write here, we can write for small perturbations. For a small region, we can write a small region at xe

$$\lim_{\|\delta x\|\to 0} \frac{h(x_e, \delta u)}{\|\delta x\|} = 0$$

Equation 4 can be written as

$$\delta\dot{x} = \frac{\partial f(x_e)}{\partial x}\delta x$$

So, this is basically a linear system, and based on the first method of Lyapunov, we have three different conditions to check the stability of the system. So, the first one is

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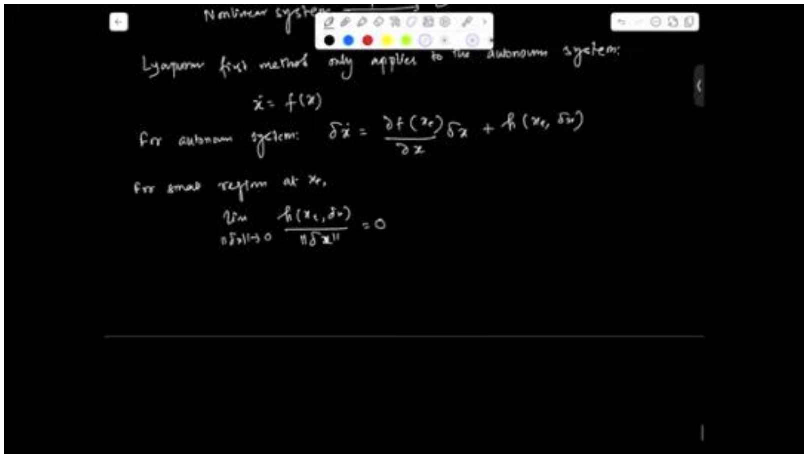

If the linear system, if the, so let us, let me write in a simplified form, we can write $\dot{x} = Ax$, for example. And here, basically, this part, we are considering as an A matrix. Now, if the eigenvalues of A have negative real parts, negative real parts. We can say the equilibrium component x_e Xe is asymptotically stable. Asymptotically stable. This is very, very important. So this is the Lyapunov first method.

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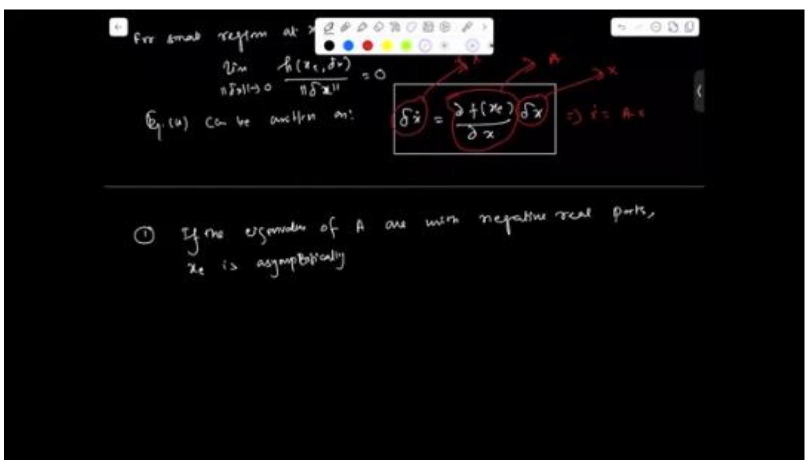

This is the first point in the Lyapunov first method. And the second point is the linearized system. Linearized system. Has one or more, it means we can say we can use like this is basically eigenvalues of A, more eigenvalues with positive real part, real part. There we can say x is unstable.

So, these two points are very, very important in Lyapunov first method. So, based on the roots or eigenvalues of the A matrix, we can comment on how the system will behave at the equilibrium point. So, if all the eigenvalues have negative real parts, then we can If it has negative real parts, then you can see the equilibrium point, the equilibrium point is stable. And if it has one or more eigenvalues with positive real parts, then we can see the equilibrium point is unstable. And the third point, there is a third point which is if the linearized system Has one or more eigenvalues with zero real parts and all remaining eigenvalues have

Negative real parts, the stability of xe cannot be ascertained. By starting the linearized system alone. You can write stability in the small. This is also an important point. It is difficult to comment if it has remaining eigenvalues with negative real parts. So, these three important points talk about the Lyapunov first method. We'll have, in the next lecture onwards, how we can come up with the Lyapunov second method and also have the Lyapunov theorem. So, I hope this part is clear. So, basically, we'll find the linear system. And from the linear system, we will see the eigenvalues. And based on the polarity of the eigenvalues, we can comment on how the system stability would be. Let's stop it here. We will continue from the next lecture.