Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 30

Limit Cycles

In today's lecture, we will be discussing these topics: phase trajectory of the circular system, limit cycle, and also we will have the MATLAB simulation of the example. This is a very important lecture; we will come up with the concept of limit cycle, which is a very powerful term in nonlinear system synthesis. If you go back to our last lecture based on the local linearization of the system. And the eigenvalues whatever we had, and based on the eigenvalues, we can come up with the conclusion circle. So, we can write based on the eigenvalues; we have different terms: one side node, circle, saddle point.And focus, so different conditions for the equilibrium point, right? So, these are the concepts. Anyways, we'll be having a lecture as well, but in extra things, we'll be doing in a literature. This is very important when this condition will arise, so let's start with the example. Let us consider the second-order system

$$\ddot{x} + \zeta(x^2 - 1)\dot{x} + x = 0$$

So, this is the second-order system. So, to study the eigenvalues problem, we can convert it to first-order ODEs. So, we can apply the change of variables. So, for that We can write

$$x_1 = x$$

$$x_2 = \dot{x}$$

$$\dot{x}_1 = x_2 = f_1(x_1, x_2)$$

$$\dot{x}_2 = -x_1 + \zeta(1 - x_1^2)x_2 = f_2(x_1, x_2)$$

and to find the equilibrium point, we have to put $f_1(x_1, x_2) = 0$ or you can write $x_2 = 0$, and from the second equation, you can write

$$f_2(x_1, x_2) = 0$$

$$-x_1 + \zeta(1 - x_1^2)x_2 = 0$$

Or $x_1 = 0$. The equilibrium point for this particular problem means the equilibrium point has 0 0. You can write

$$X_e = \begin{bmatrix} X_{1e} \\ X_{2e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now, we will find the Jacobian matrix. The Jacobian matrix we can form as

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_2}{\partial x_1} = -1 - 2x_1x_2\zeta, \quad \frac{\partial f_2}{\partial x_2} = \zeta(1 - x_1^2)$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2\zeta & \zeta(1 - x_1^2) \end{bmatrix}$$

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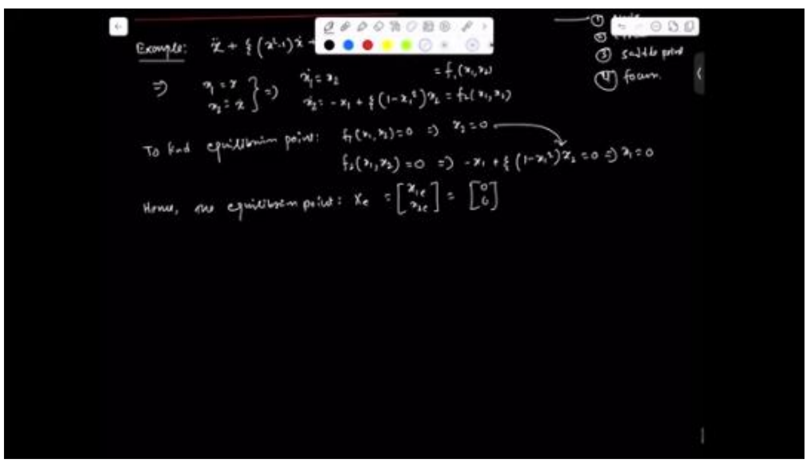

At (0,0)

$$J = \begin{bmatrix} 0 & 1 \\ -1 & \zeta \end{bmatrix} = A$$

This is our Jacobian matrix at 0, 0 equilibrium point. Now, if you want to find the eigenvalues, we have to find the characteristic equation of this J matrix. This is also sometimes called the system matrix in defined. So here, we can write

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \zeta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And this is equal to the linearized system. This process we have done multiple times in this course. Here, now we can find this is the matrix we can get, say. So, from this diagonal values,

$$|\lambda I - A| = 0$$

and the characteristic equation will have

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda - \zeta \end{vmatrix} = 0$$

$$\lambda(\lambda - \zeta) + 1 = 0$$

and the roots or eigenvalues, roots or eigenvalues, we can find

$$\lambda_{1,2} = \frac{\frac{\zeta}{2} \pm \sqrt{\zeta^2 - 4}}{2}$$

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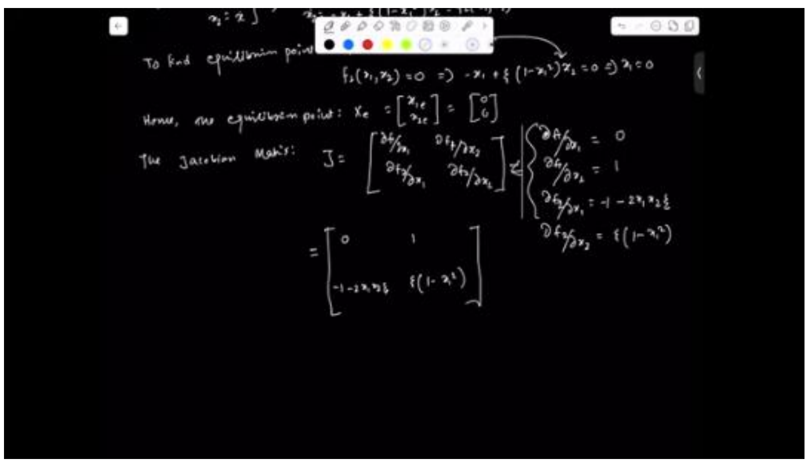

If you notice carefully, if you notice carefully, for the different values of ζ, we have different eigenvalues, right? So, let us look based on, based on the range. The range of ζ, the eigenvalues, eigenvalues are classified, classified as at ζ equal to 2. If you notice carefully, at ζ equal to 2, we are getting. Eigenvalues which are basically one. So, this we can say, eigenvalues, we can say eigenvalues are same and positive. So, what is magnetic? When the eigenvalues are equal, not distinct, and there's, of course, both are positive. So, this is special in this problem, and this actually creates the limit cycle problem. So, let's do the other values. I'll come, I'll come to this point again. So, let us again, if ζ is greater than 2, in this case, if you notice. $\lambda_{1,2}$ distinct and positive, right? That is obviously if we come of this condition in this expression, we will have distinct and positive, that is clear. And let us have, if ζ is less than 2, in this case, the eigenvalue should be Complex conjugate with ideal part with a

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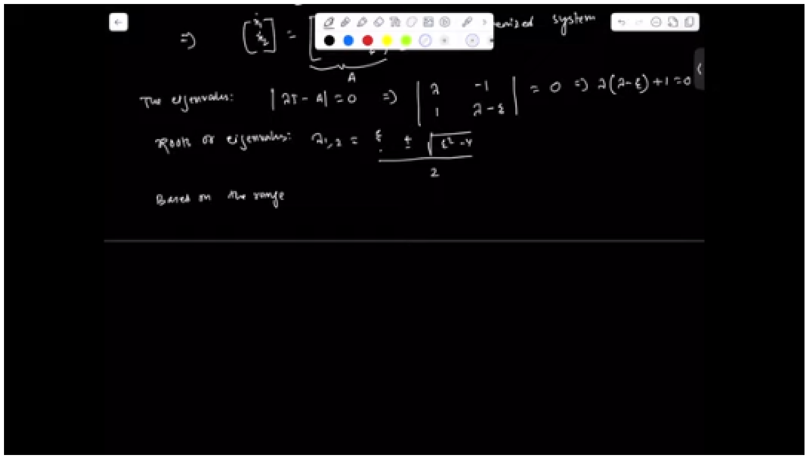

with will have positive real part of code here. Complex with positive, if theta equal to zero, in this case, eigenvalues are basically equal to zero. Then it will be only we will have complex conjugate eigenvalues, but real part will be 0. So, in this case, $\lambda_{1,2}$ will be complex conjugate with the real part. Right.

So, this is very, very important conditions for this particular problem. Now, if you come up with the now different scenario. So, here let us assume here we have different trajectory for. For this ζ value which, which varies from minus λ to, for example, minus infinity to plus infinity, for example, sorry, let us, let us assume minus alpha to plus alpha, the ζ varies. So, in that case, we can have different scenario when the, when the ζ varies. From minus or we can write when ζ is less than zero, in this case, if you notice here, if it is less than zero, so here we will have this is negative part, but this part will be complex conjugate, this part will be complex, right? So, in that case, if you remember what you have done in the last lecture, so in that case.

We will have stable focus, right. So, in that case, we can plot something, we may have plotted something like this. So, it is converging to 0, right. This is what we have done in the last lecture. And if ζ is equal to 0 for this case, in this case, we can write

So, these are the conditions we can assume for the special cases because since ζ is 2, then we can come up with easy conditions for the eigenvalues, and this is what actually happens if it varies from minus to plus. So, now if ζ is equal to 0 in this case, so what will happen if it is 0? Then only we will have this part, right. It will be complex conjugate with a real part of 0. So, in that case, what you have studied will have a center, right, or vortex. So, these trajectories will be something like this for the different conditions; we will have different circles.

So, this is also called a vortex. So, this is a stable focus. And this is a vortex or circle, or we can say a center also, right? Now, if ζ is greater than zero, let us look. If ζ is greater than zero in this case, if ζ is greater than zero, then obviously we may have this as complex conjugate, maybe. But here, actually, we will have positive values, right? So, in that case, we can say, yeah, this may actually be diverging, right? Diverging. So, this is the condition for the z term. Is greater than zero. So this, and it is actually, we can say stable or unstable, of course. This is something we have already done. Now the case is this case. This is very important. So this actually creates a problem of a limit cycle.

So here, we are not guaranteed. It is difficult to conclude whether the trajectory starts or where it has to end. So, let me say something here. Let me write a note for this particular thing; it is very, very important. There is, before that, a question.

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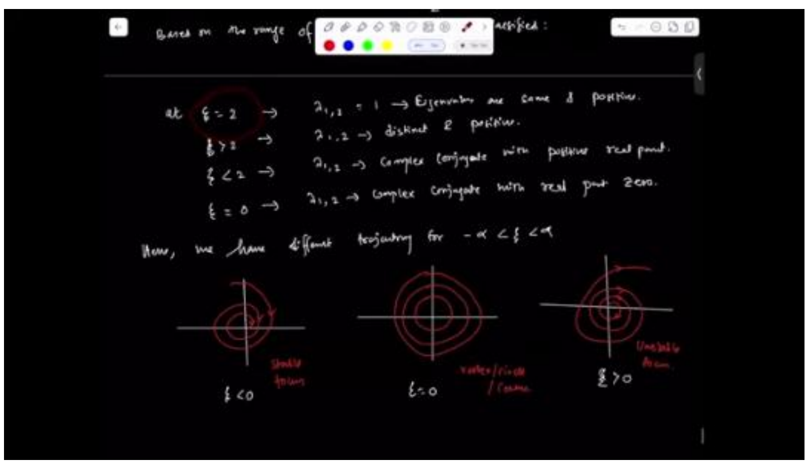

For ζ greater than 0, for ζ, we will use this. This is something also called an orbit. These are the orbits, maybe. These are the orbits, how the orbits will happen. So, this kind of thing is well required for studying orbital mechanics. So, here, will this orbit, will this orbit go to infinity? So, it means, for this particular condition, this trajectory will go to infinity. This is the question, actually.

Now, the note is there is no guarantee because we derived our remarks based on the local linearized local linearization or linearized for a small part of the region for a small part of the region you can write. So, this is what it means because if you notice our linearized system, we obtained this is basically For the small perturbation around the equilibrium point, how the system behaves, right.Actually, if you write more practically, if you want to write this expression, we can write this expression, it should be

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \zeta \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

These are the perturbed variables, right. The actual equation is basically the perturbed system, right, and we are studying how these perturbed variables are going to behave over time. This is actually what we are doing in the state-to-state trajectories. So now let me go through the note again. Thus, it is difficult to comment that it will continue to diverge. Maybe after some time, because this conclusion we derived based on the local linearization, maybe after some time that may again converge, right? So that's why it is difficult to comment. If we go away from the singular point or equilibrium point, there is no guarantee that the local linearization will be valid. That is obvious. Maybe after some time, you can write after some time the orbit

Still incoming from the towards you can write towards the equilibrium point. So, based on this comment, we can come up with some kind of phase portrait, something like this. It is very, very important, something like this. It may be, may be for some condition of, for some condition of ζ , we can have the unstable focus but some other condition maybe we can have something like this something like this maybe it is incoming one and so it means in between there will be some different orbit which will be stable so it means we can write here very very important comment in between these orbits there will be some orbit which will be stable in both sides right both sides. So, this is how it means we can comment like in between this there can be another orbit which can be something like this which will be stable between them and this is how limit cycle occurs and this kind of situation we call if this kind of orbit arises in the system it is called stable limit cycle. For the different, if the orbit is perturbed, if the orbit is perturbed, for example, in some of the orbits perturbed, and if you have, for the perturbed, if the perturbed, if you have different kind of circle like this, because this, this case, this particular, this case, it is part at this point, right, but it is, maybe it is not perturbed. another orbit and this kind of orbit is called it is called Unstable Limit Cycle. So, this is how the limit cycle occurs in the system.

So, it is difficult to come and show in particular for our example this is the condition actually is gives the limit cycle in the system. Now we will have the MATLAB code for this particular example. If you notice here we have the equations, we have the two equations here and this is the dot and we have used the ODE 45 to solve this nonlinear equations. And if you notice here x_1, x_2 they are also actually sustain oscillation is there and throughout the orbit and if you notice here this is actually limit cycle and this is our origin this origin and about this this is something happening like this so what you have explained here something here same thing is happening here and this is actually limit cycle for this particular example

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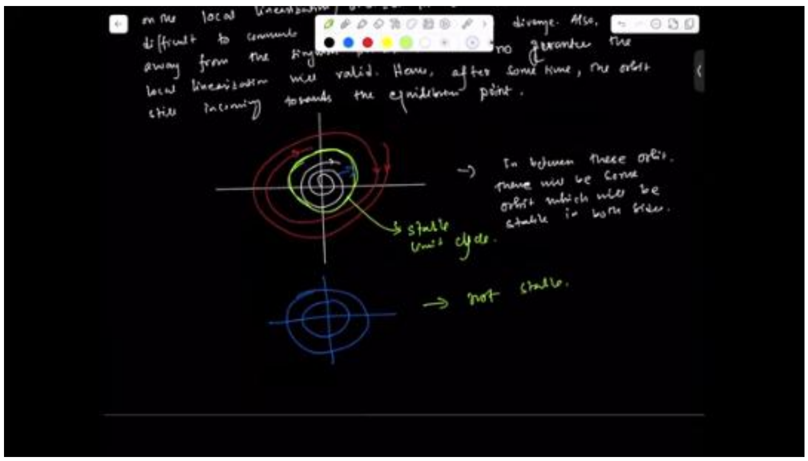

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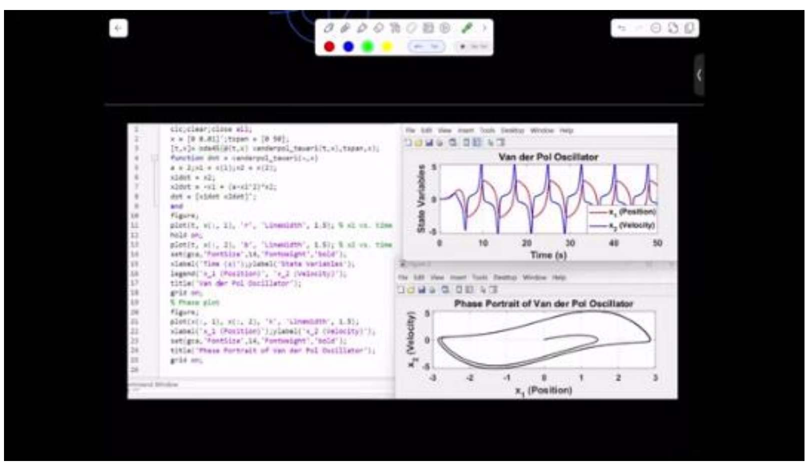

And in this example, this MATLAB code is for this particular example. But here, if you notice, we have taken this example where we have set a equal to two. Now, you can try in MATLAB, in your MATLAB window, you can try. You can try placing, um, you can take if a is less than two and a is. So, these two conditions you can please try. So, maybe you can take a equal to 1 and a equal to 10. So, how you can get this kind of response, you are supposed to get for one case, you are supposed to get this and this. So, one condition is supposed to have this, another condition is supposed to have this.

So, in the last lecture, we had many examples. So, you can follow the same procedure on how to find the phase portrait, one variable with respect to another variable. So, this is x_1, plotted with respect to, sorry, x_2 plotted with respect to x_1. So, this is the same way you can follow to write the MATLAB code for the other examples that you have done in the last lecture. I hope this will help you out to understand the limits.

To get the phase portrait of the second-order system, and we will stop it here. We will continue from the next lecture. In the next lecture, we will come up with the Lyapunov part, how we can analyze the system using the Lyapunov function. Thank you.