Advanced Aircraft Control Systems With MATLAB / Simulink

Prof. Dipak K. Giri

Department of Aerospace Engineering

Indian Institute of Technology Kanpur

Lecture 03

State Space Equation and its Solution

Hello everyone, this is the third lecture in this course. In today lecture, we will be discussing how we can find the solution of the system, which is defined in the state-space form in terms of state-transition matrix. State-transition matrix is a very powerful term which actually finds the solution of the system from any initial time to any final time, it can be any value. and also it separates the internal dynamics of the system from the influence of the external inputs, so this is very very powerful term and where you can study the long-term behavior of the system and also we'll find the solution both in scalar and vector case, it means first we'll take the scalar system, we find the solution which will be followed by the finding the solution of the state and the system which is defined in the state space form, so let's start the lecture, first we'll take the scalar system and we'll find the solution of the scalar system, so these are the

topics we will go over in today lecture. So, here let us start with the system. We have the system, scalar system

$$\dot{x} = ax(t) + bu(t) \ldots . Eq(1)$$

this is scalar system and here let us find the solution for t greater than t naught. So, t now we can assume by the simple time value and t can be any value and here what we are going to do is also let us assume the initial state starting from t naught which is denoted as x_0.

Now, if you multiply both side equation 1 by $e^{-a(t-t_0)}$ we get

$$e^{-a(t-t_0)}\dot{x}(t) - e^{-a(t-t_0)}ax(t) = e^{-a(t-t_0)}bu(t) \ldots Eq(2)$$
$$\frac{d}{dt}\left[e^{-a(t-t_0)}x(t)\right] = e^{-a(t-t_0)}\dot{x}(t) - ae^{-a(t-t_0)}x(t)$$

From Eq(1)

$$\frac{d}{dt}\left[e^{-a(t-t_0)}x(t)\right] = e^{-a(t-t_0)}bu(t) \dots Eq(3)$$

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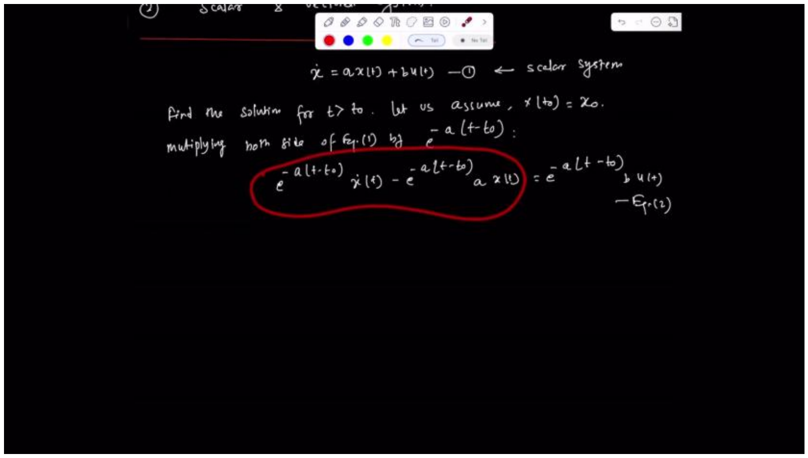

I actually noted that in most of the aerospace application, we assume most of the aerospace application, we can say We assume $t_0 = 0$.

$$\frac{d}{dt}[e^{-at}x(t)] = e^{-at}bu(t) \dots Eq(4)$$

Now, if we apply the convolution theorem to this equation number 4, we can write from this equation, if we integrate equation 4 from t equal to 0 to t, we can write

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau$$

$$\Rightarrow e^{-at}x(t) = x(0) + \int_0^t e^{-at}bu(\tau)d\tau$$

$$\Rightarrow x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau \dots Eq(5)$$

So, this is very very important solution of the differential equation what you have considered.

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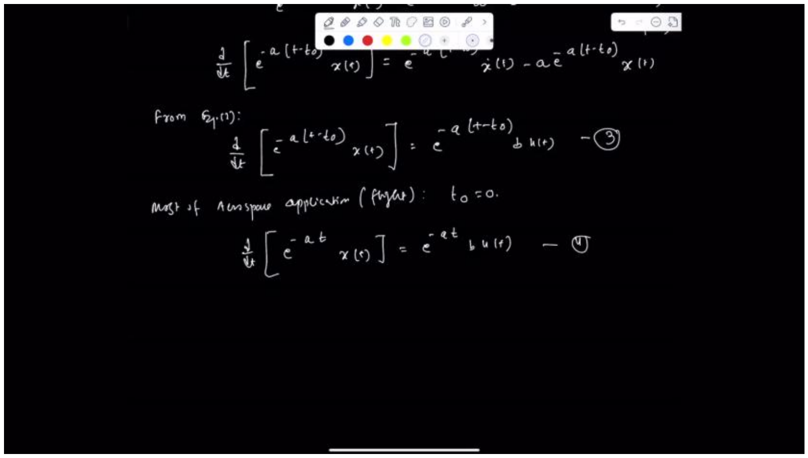

So if you look closely this equation, this term actually talks about the natural system behavior because here if you notice here we can assume this is the term coming into the natural dynamics and this is the term due to the control input of the scalar system. So the solution of this this equation of x(t) This talks about the natural system behavior and this is talks about the how the external input is going to behave the system over time. So, we can say, so here this integral term talks about how the system will behave in the steady state.

So, we can write here the integral term talks about the system behavior at steady state. And this term we can say it is more of a system is going to go through the transient behavior. So, this is how we can find the solution of the scalar system. Now, let us extend, since we are going to deal with the state space system, so now let us find, how can we find the solution of the state space equation.

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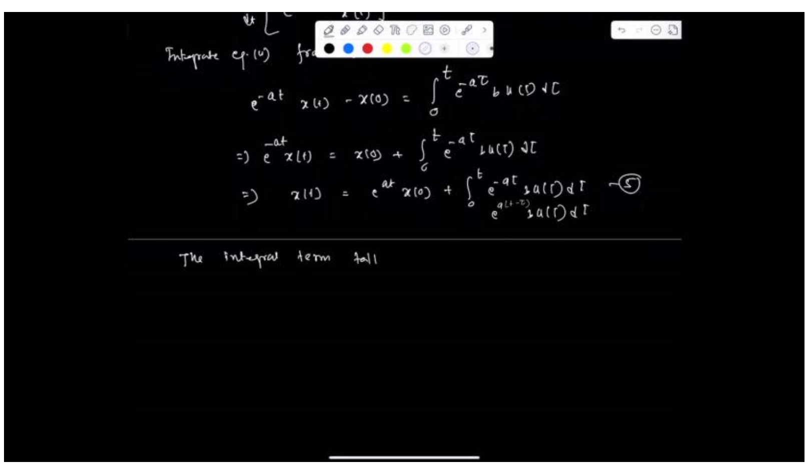

So, let us work on that. Now, let us assume if the system is defined as

$$\dot{x}(t) = Ax(t) + Bu(t)$$

So, this is the state space equation. So, here one thing, this is the vector, this is matrix, this is the vector, this is also matrix, this is the vector. So, this system is defining state space problem.

Now, let us find the solution of the this state space equation, this is the main goal of this lecture, how we can find the solution of the system which is defined in the state space form, so now first let's consider since the system is quite complex because the control input also in terms of the vector, so first let's go with the without influence of the control input, how we can find the solution of the natural system, so let's work on this without the influence of control input, let us define this equation number one without the influence of control input, we can write the equation one,

$$\dot{x}(t) = Ax(t)$$

Now, if you apply Taylor series for the x(t) expression, we can write, Taylor series is also something called power series. The power series, assume that the power series of X(t) can write, so it is actually capital X,

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots .$$

$$\underline{b}_1 + 2b_2 t + 3b_3 t^2 + \cdots + kb_k t^{k-1} + \cdots = A(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k +) \ldots Eq(4)$$

$$b_1 = Ab_0$$

$$b_2 = \frac{1}{2}Ab_1 = \frac{1}{2}A^2 b_0$$

$$b_3 = \frac{1}{3}Ab_2 = \frac{1}{3 \times 2}A^3 b_0$$

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$$b_k = \frac{1}{k!}A^k b_0$$

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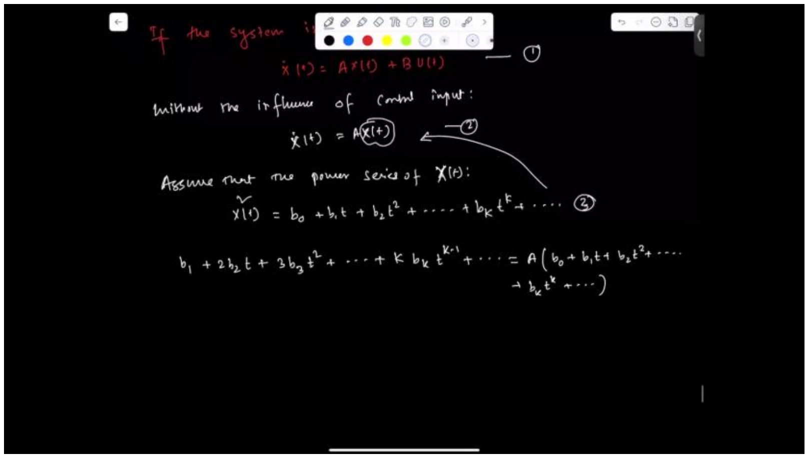

So, this is the quantity we are getting from equation number 4, and so now also we can put since we are going to t equal to 0 for this particular case, so we can write in equation two, if you put t equal to zero, also you can write for equation, now we will put a t equal to zero in equation three and find the initial value of the state of t equal to zero, so let's say at t equal to zero, from equation three we can write $X(0) = b_0$. Now, if you substitute $b_1 b_2$ till b_k in equation 3

In place of x0, we can substitute b then we can get our expression is what we're going to do is, we will substitute all these terms in equation three, so we can write x of t we can write

$$x(t) = \left(I + At + \frac{1}{2!}A^2t^2 + \cdots + \frac{1}{K!}A^k t^K + \cdots\right)x(0)$$
$$x(t) = e^{At}x(0) \ldots Eq(5)$$

So this is the solution of the system without control.So this is the solution of equation 2. in the absence of control vector matrix, so now, if you remember from the mathematics, what you have done before this is nothing but the expression of e^{at}, so from this expression, so this is the Taylor series of e^{at}, so now this from this solution it is clear that we can transform any value x naught to x t using the state transfer matrix, this is very very important, so this state transfer matrix is very very useful how we can transform from x_0 to x t using e^{at}, matrix and this term we call state transition matrix, this is very very powerful term, how we can transform from one state to another state using this, this is in the modern control engineering concept, we call it state transition matrix, so now our next part is, if you take control input in a picture, so how the system solution would be looking like, so let's work on this, now solution of state

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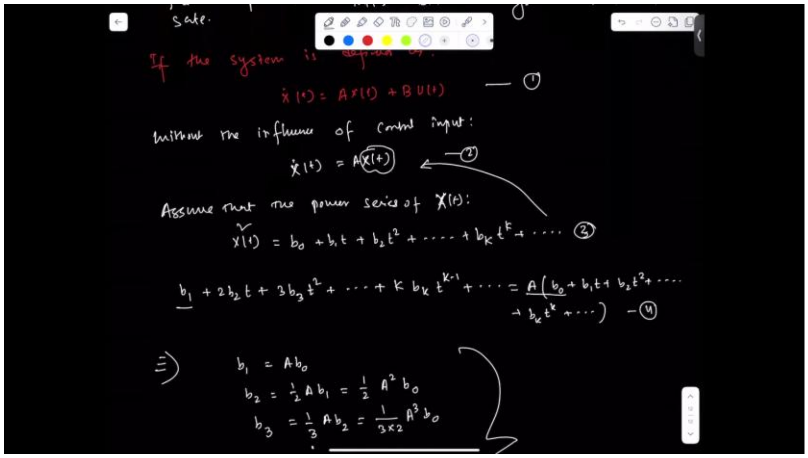

equation including the control input. Initially we have considered there is no control input, now we are going to assume control input. So, in this equation there was no control input, now we are going to consider the control input into the state equation. Now, if you consider the control input here, the control input write here u(t), so now we can write, the state equation

$$\dot{X} = AX(t) + BU(t) \dots Eq(6)$$

so let's assume u(t) is not equal to zero and we'll find the solution t plus three, we get to be equal to t naught, okay, so we will follow the same procedure what we have done for the scalar system case, so we will multiply on both side. Now, if you multiply equation 6 by a term, e^{-At}, multiply, same procedure what we have done for the scalar system. we can write

$$e^{-At}[\dot{X} - AX(t)] = e^{-At}BU(t)$$
$$\frac{d}{dt}[e^{-At}X(t)] = e^{-At}BU(t) \dots Eq(7)$$

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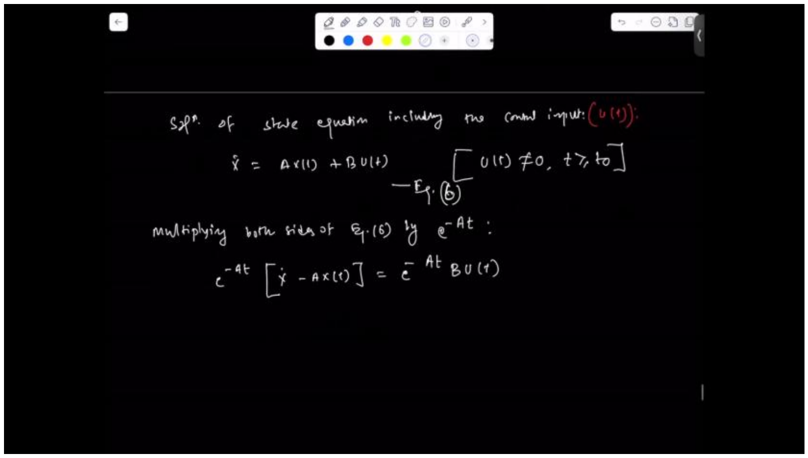

So, here integrating integrating equation 7 from t equal to 0 to t, we get

$$e^{-At}X(t) - X(0) = \int_{0}^{t} e^{-A\tau}BU(\tau)d\tau$$

$$e^{-At}x(t) = x(0) + \int_{0}^{t} e^{-A\tau}BU(\tau)d\tau$$

$$x(t) = e^{At}x(0) + \int_{0}^{t} e^{-A(t-\tau)}BU(t)d\tau \ldots Eq(8)$$

So, in the absence of control input, we already have this system, right, this system, but if we have the control input, we are getting this extra term in our solution. So, now, this is very, very important here. So, if we assume all the eigenvalues of A matrix are negative, we can say that the system x of t goes to 0, as time tends to infinity, can write, in the absence of u(t) if eigenvalues of A are negative x of t goes to 0 as t tends to infinity. But due to this term, it is difficult.

So, now, so similar to the previous case, this is the term which actually making the steady state system not to track the desired values. So, this is how we can find the solution of the state space system in vector case and I will show you the solution for the scalar case as well. Now, how we can find the solution of the state space system in terms of Laplace domain. Let us work on this.

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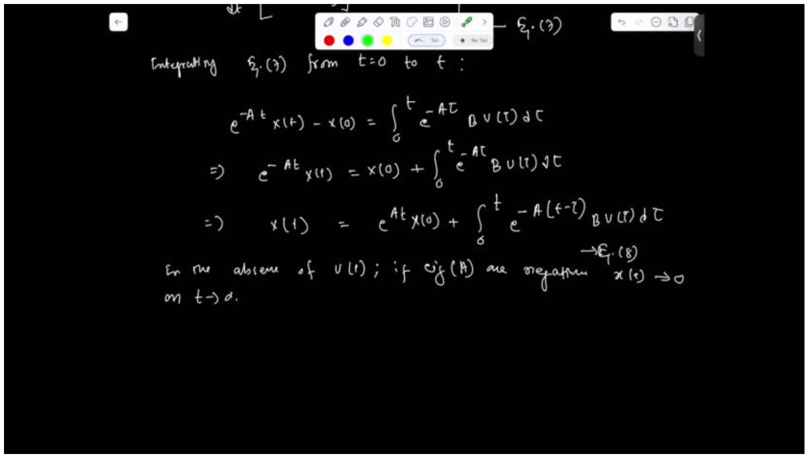

Solution of state space system, in Laplace domain. So, here our system is

$$\dot{X} = AX + BU \ldots Eq(1)$$

So, now taking Laplace transform we can write

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$\Rightarrow SX(S) - AX(S) = X(0) + BU(S)$$
$$\Rightarrow (SI - A)X(S) = \times (0) + BU(S)$$
$$\Rightarrow X(S) = (SI - A)^{-1}X(0) + (SI - A)^{-1}BU(S) \ldots Eq(2)$$

please note, there is important formula also for SI minus A, whole term inverse, so once we know that this SI minus A inverse I

$$(SI - A)^{-1} = \frac{I}{S} + \frac{A}{S^2} + \frac{A^2}{S^3} + \cdots Eq(3)$$

we are directly using but if you want to know how these are coming, please refer to some book because if you are going to detail of it it will take time lot of time so now if we take the inverse Laplace transform, if we take the inverse Laplace transform of this whole equation we can write inverse Laplace transform, we can write

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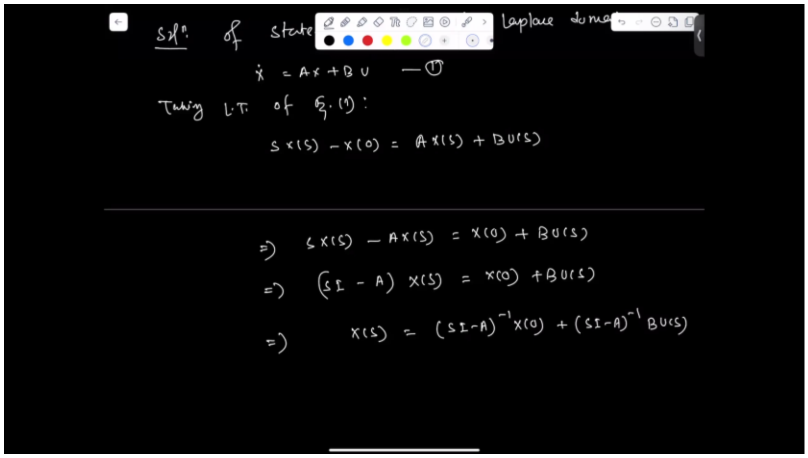

$$\mathcal{L}^{-1}\{(SI-A)^{-1}\} = I + At + \frac{A^2t^2}{2!} + \frac{At}{3!} + \cdots$$

$$\Rightarrow (SI-A)^{-1} = \mathcal{L}\{e^{At}\}$$

$$X(s) = \mathcal{L}\{e^{At}\}X(0) + \mathcal{L}\{e^{At}\}BU(s) \ldots Eq(4)$$

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}BU(\tau)d\tau$$

So, this is the solution if you apply inverse Laplace transform, also you can find the same expression, what you have done in the time domain analysis. So this is here very, very important term. The main result of this lecture is how we can find the state transformation matrix, which can help us to derive the solution of the system from any initial state to any final state. And if there is control input in our system, how we can use the state transformation matrix to find the solution that as well.

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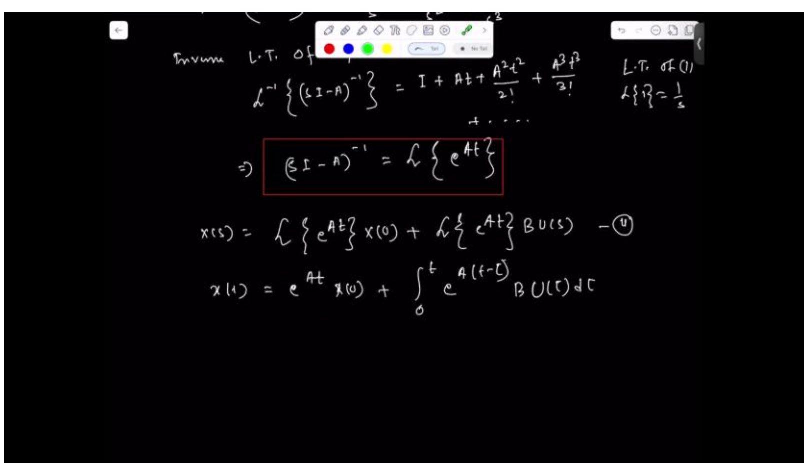

So, let us stop it here. We will take an example in the next lecture on how we can validate whatever we have done in today's lecture. Thank you.