Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 29

Classification of equilibrium points of planar nonlinear systems

In the last lecture, we came up with the equation in this form, and this is the solution of this system. It has a system in it that is in decoupled form, and from the decoupled equation, we can write the decoupled equation dynamics. These are the solutions of the decoupled system, and this is the closed-loop closed form of the system

$$z_2(t) = cz_1(t)^{\frac{\lambda_2}{\lambda_1}}$$

So, in today's lecture, we will define the equilibrium point of the system of the original system. What we had before is the original system, and for this original system, how we can define the equilibrium point, whether it is a node, focus, saddle point, or vortex. This is a very important topic: how we can classify the equilibrium point. So here, we will come up with the different conditions for the equilibrium for the eigenvalues. So, case one, we are continuing from the last lecture. So, the first condition we can let us assume is that the eigenvalues are real.

So, under this condition, if the eigenvalues are real, we have the first case, maybe a subheading. If the eigenvalues are real, not complex, real eigenvalues and distinct, meaning they have different values, they are not the same, and negative, if the eigenvalues are negative. So, the second-order system We are supposed to have two eigenvalues which are already there, lambda 2 and lambda 1. If the eigenvalues are real and negative, so in that case, we can write

$$\frac{\lambda_2}{\lambda_1} = K \geq 0$$

$$z_2 = cz_1^k$$

so it will be some constant because both are negative. so this is the equation for the parabola that you know. This is the equation for the parabola. So, for this particular condition, if you plot z_1 versus z_2 , so, for example, so the trajectory, the state trajectory, will be something like this, so it is converging to the equilibrium point 0, 0. So, this is what we can say, and so it means if the eigenvalues are real, distinct, and negative, then state conversions will look like this. And this condition we call, if you have the trajectory like that, it is called a saddle. Stable node, this is actually a stable node, this is a stable node. So, this is the first case. We can try MATLAB, you will have something like this.

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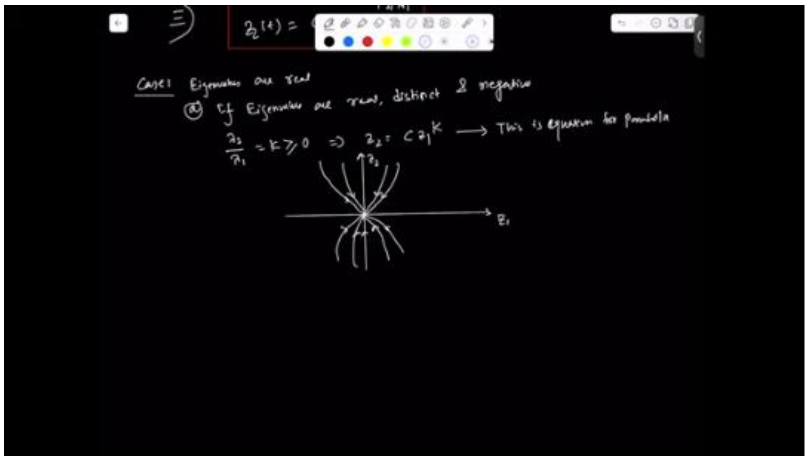

And the second case, if the eigenvalues are real, distinct, and positive, if the eigenvalues are positive, then in that case. So, for example, $\lambda_1, \lambda_2 > 0$ So, it means you can write $\lambda_1 \neq \lambda_2$ because it is also mentioned distinct here, right? So, if you have this condition for the eigenvalues, the state trajectories will be something like this. so the state trajectory will be diverging from the equilibrium point. It is not converging, so this is what we call an unstable node, an unstable node. And if the trajectories are like this, then it is called a node. And if it is converging to zero then it is stable. If it is diverging from zero, then it is unstable. It is not converging to zero, that is why it is unstable. The third case, the third case, eigenvalues are real, distinct, one positive and the other negative. Okay, so in the previous case, both are positive and distinct. In the first case, both are negative and distinct. And in this case, one is positive, one is negative, but distinct. In this case, the trajectory will be this is z_2.

So, it will be something like this. This can be the state. Called a certain point, so if you want, you can do it in MATLAB. You can easily simulate this equation in MATLAB, and you can take the different values of λ_2 and λ_1, and you can plot it. So, this is the case if the eigenvalues are distinct. In this case, if it is real and destined to have the A, B, C condition, and if the eigenvalues are complex, so in this case. It is basically under this, also having some condition.

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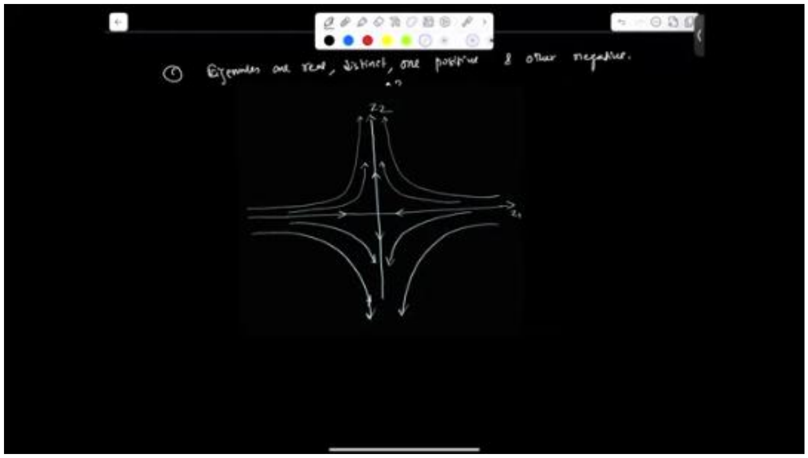

First, the eigenvalues are complex with a negative real part. So, here, basically, we can assume. If they are

$$\lambda_{1,2} = -a \pm ib$$

So, we have the real part as negative, and in this case, the straight trajectories will be z_1, z_2. So, it will go like this. So, it is converging to the equilibrium point. There is the equilibrium point, and the system is stable, basically. So, in this case, it is called a stable focus, OK? And second, under this section, if the eigenvalues are complex. With a positive real part, we will take an example. It will be clear what we are doing here. So, in this case, it will be an unstable focus. Obviously, it will diverge from the equilibrium point. So, in this case, we can start moving here. It will go like this, so it is basically. Like z_1 , z_2, and this is basically an unstable focus. So, it means if the eigenvalues are complex, if there is a negative real part, then it will be a stable focus. If it is a positive real part, then it is an unstable focus. Just try to understand what is happening here.

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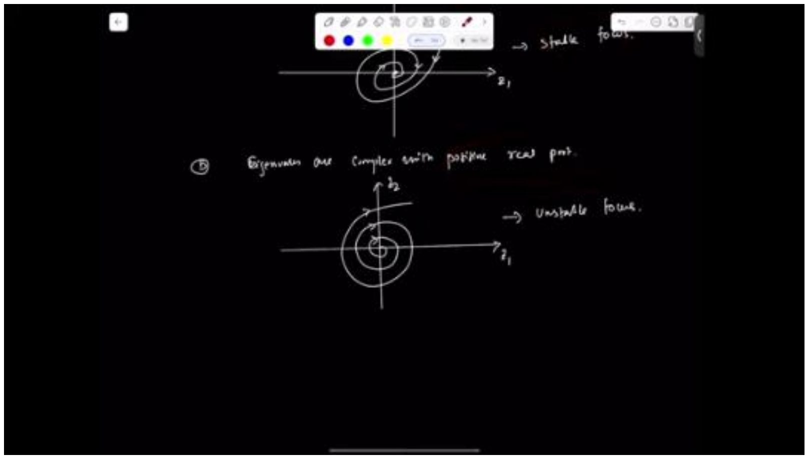

This thing will be important in later lectures, and in part C, eigenvalues are complex with, sorry, with 0 real part, with 0 real part. It means $\lambda_{1,2} = \pm ib$ with no real part. So, in this case, the state trajectories will be something like this, it will be The trajectories are z_2 and z_1 , so it is something like this, and this kind of condition we call a vortex or center. This is how we can define the equilibrium point of the system. So, based on the equilibrium point, we are coming to this conclusion, right?

Because the eigenvalue is being found from the linear system, and the linear system is linearized around the equilibrium point. And based on the eigenvalues, how the system trajectories move in the space, in space, and we can call it like that. Now, let us take an example; then the problem will be very much clear. Let us take an example. So, we will start the system from the nonlinear system and see how it can get linearized, and then we can study the system behavior. The question is to draw the state trajectory of the system.

$$\ddot{y} + 0.5\dot{y} + 2y + y^2 = 0$$

So, if you notice in this equation, this is the nonlinear term, the square term, right. Other terms are linear. So, this is the nonlinear term. And due to which the system is nonlinear. And also, also Analyze the equilibrium points, equilibrium points. So, for this, we have to first find the linear system, find the equilibrium eigenvalues, and based on that, we can define or classify the equilibrium point whether it is stable, unstable, yeah, solution. So, let us define the system in state variable form. So, let us define

$$x_1 = y$$

$$x_2 = \dot{x}_1 = \dot{y}$$

$$\dot{x}_1 = x_2 = f_1(, x_1, x_2)$$

$$\dot{x}_2 = -2x_1 - x_1^2 - 0.5x_2 = f_2(,x_1,x_2)$$

So if you notice this equation, this system, actually there is no control, the natural system and we are studying the system, whether it is stable or not, yeah. Now to find the equilibrium point, we know how to find, we will put, we will make this equal to 0 and this equal to 0 and find the values of x_1, x_2 and this will give us the equilibrium points. So let us do to find the singular points or equilibrium points or fixed point in the different terminologies fixed points so we can write

$$f_1(,x_1,x_2) = 0 \rightarrow x_2 = 0$$

$$f_2(,x_1,x_2) = 0 \rightarrow -2x_1 - x_1^2 - 0.5x_2 = 0$$

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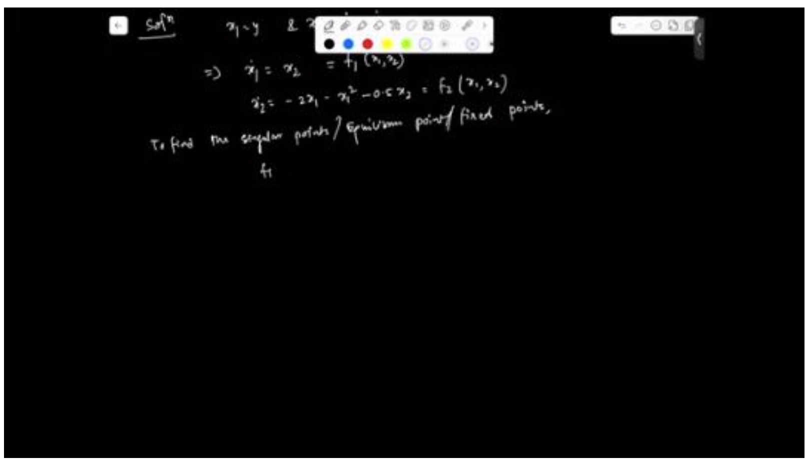

So, if you substitute here x_2 equal to 0, we have the equation

$$x_1 = 0 \quad or \quad x_1 = -2$$

So, the equilibrium conditions are, or equilibrium points are, the equilibrium points are, we have $(-2,0)$ and $(0,0)$. So, for this nonlinear system, we are having two equilibrium points, right.

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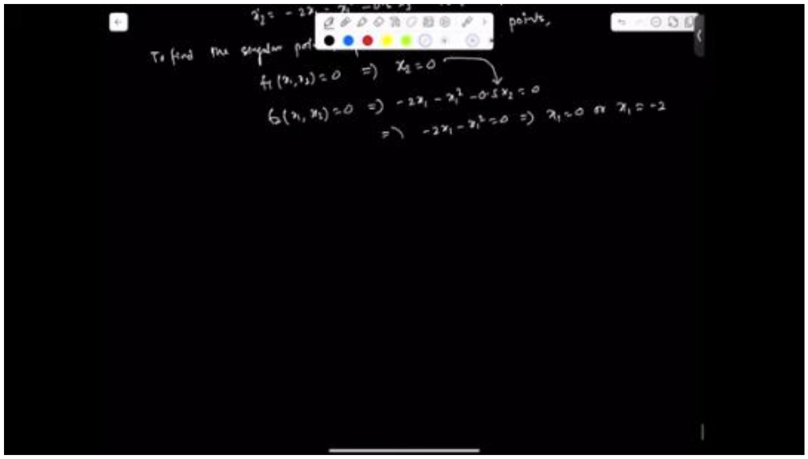

So, we have to linearize the system about these two equilibrium points, so we need to find the Jacobian matrix. So, the Jacobian matrix we can find. The Jacobian matrix we can write as J equal to, we can write

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

So here, f_1, f_2 is nothing but our two functions. This is one function, and this is another function, right? So here, let's find. So here, we can find

$$\frac{\partial f_1}{\partial x_1} = 0, \ \ \frac{\partial f_1}{\partial x_2} = 1, \ \ \frac{\partial f_2}{\partial x_1} = -2 - 2x_1, \ \ \frac{\partial f_2}{\partial x_2} = -0.5$$

Now, if you substitute these expressions, these expressions, if you substitute this one here in the Jacobian matrix, you can write

$$J = \begin{bmatrix} 0 & 1 \\ -2 - 2x_1 & -0.5 \end{bmatrix} = A$$

And so, this is, you see, if you write in linear system form, we can write the linearized system. We can write

$$\partial \dot{X} = \begin{bmatrix} \partial \dot{x}_1 \\ \partial \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 - 2x_1 & -0.5 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

So now here, basically, we can write, this is also you can even say A matrix, this is also defined as A matrix. So now let's find the A matrix.

The first equilibrium point is (0,0). We have two equilibrium points: (-2,0) and (0,0.8). Let's start with the first (0,0) equilibrium point. For that, we have the A matrix as

$$A|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -0.5 \end{bmatrix}$$

and the eigenvalues. To find the eigenvalues, you can solve $|\lambda I - A| = 0$. If you substitute the values, we get

$$\lambda_{1,2} = \frac{-0.5 \pm i\sqrt{8 - 0.25}}{2}$$

So, if you notice here carefully, the eigenvalues

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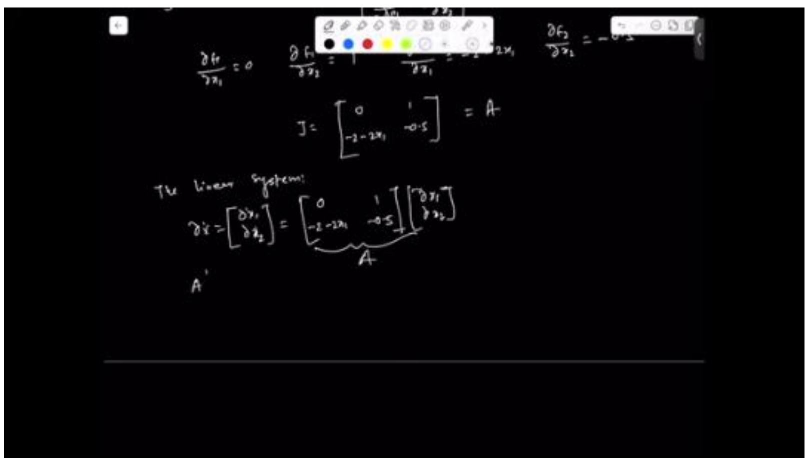

are real and they have complex conjugate terms. So, you can write here that the eigenvalues are complex with negative real parts. So, this condition, if you go back to our lecture, this is the condition we are having. The eigenvalues are complex with negative real parts. So, we are supposed to get this kind of condition. So, here, if you plot, we can have this as For example, x_1, x_2, and we are supposed to get this kind of response. And this is actually nothing but a stable focus.

So, in this case, if you notice, this is the equilibrium point. Don't forget that. That is very important. So, at this equilibrium point, the trajectory is converging to 0, 0, right? And this is a good stable focus. Now, let us work on the different equilibrium points.

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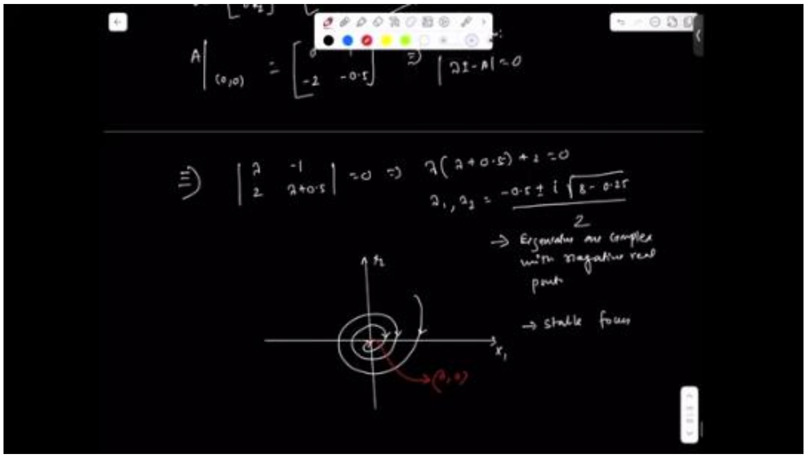

Now, let us evaluate the eigenvalues. Let us evaluate the eigenvalues. At the equilibrium point, equilibrium point minus 2 and 0. So, here we can directly write lambda I minus A at equilibrium point minus 2, 0, and if you substitute, we are having I am writing the final expression here. So, equal to 0, we can write. So, here from this, you can have the characteristic equation

$$\lambda(\lambda + 0.5) - 2 = 0$$

$$\lambda_1 = 1.19, \quad \lambda_2 = -1.69$$

One eigenvalue is positive, one eigenvalue is negative, and they are distinct. So, this is the case where eigenvalues are real, distinct, one positive, and one negative, right? So, we are supposed to get this as the condition for a saddle point, right? We can draw the final picture. So, here we have the equilibrium point zero, zero. Another equilibrium point is minus two and zero, right? So, here we are getting the stable focus. So, you are getting something like this. So, this is the

Converging to the equilibrium point, and another case here we are having the straddle point. So, this is the saddle point. So, this is how we can come up with the state trajectory of the system, and you know, While we are doing the MATLAB, it will connect maybe here, connect like this, so something like this, all will be connected, and it will get some, some formula inside this, this will be there, it will be there, so it can come up as a nice figure in MATLAB. If time permits, in the next lecture, we have some ideas for this.

For some ordinary functions, how you can find the state trajectories of a system. Now, I will take another example to make your, so, example, another example, maybe you can get a better picture of how it can come up. So, the state trajectory of a nonlinear system. So, let us consider the following system

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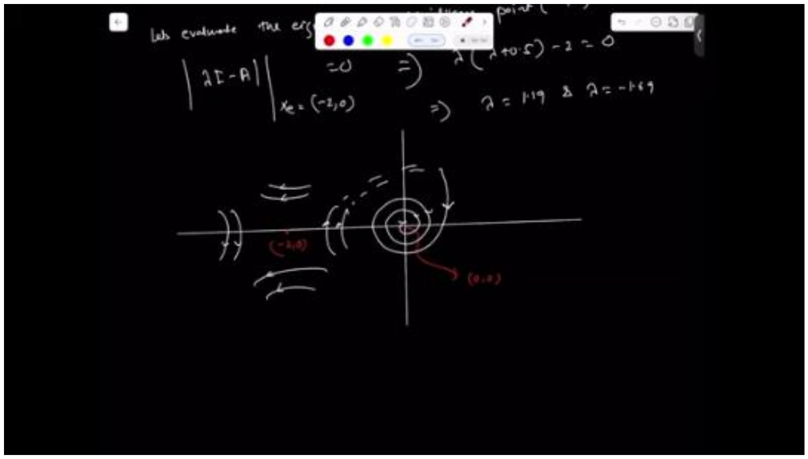

$$\ddot{y} - \left(0.1 - \frac{10}{3}\dot{y}^2\right)\dot{y} + y + y^2 = 0$$

this is a very highly complex system, right? So, here, study the system and classify the equilibrium points. So here we can, first, let's find the equilibrium point. For that, we have to, say, for a second-order system, we have to come up with the two first-order ODEs. So the solution, I'm just going through some step jumps. So, we can come up and define the state of

$$y = x_1$$

$$\dot{y} = x_2 = f_1(x_1, x_2)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + 0.1x_2 - x_1^2 - \frac{10}{3}x_2^3 = f_2(x_1, x_2)$$

And to find the trim point, we have to put f_1 equal to 0 and f_2 equal to 0, right? So, doing that, we have

$$x_2 = 0$$

$$= -x_1 + 0.1x_2 - x_1^2 - \frac{10}{3}x_2^3 = 0$$

$$x_1 = 0, -1$$

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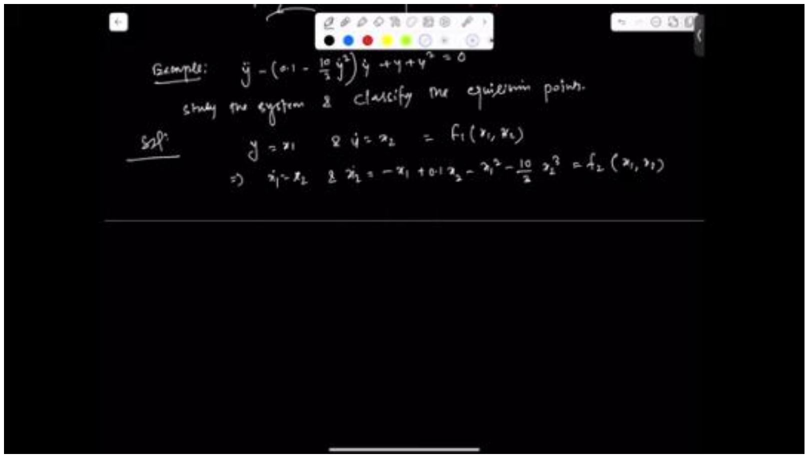

And from this, we have x_1 equal to 0 and minus 1. So, we are having two equilibrium points. Minus 1, 0 and 0, 0, these are the points. And this is, I request you, please find the Eigen values and Eigen values of the system and find the state trajectory. So, we are supposed to get the response something like this.

We can verify. So, the response will be something like this; it will be diverging from the equilibrium point, this will be 0, 0, and we have another equilibrium point, maybe minus 1 and Saddle point again. So, this is the saddle point, and this is, we can say, this is the unstable unstable focus. So, this is basically an unstable focus because it is diverging from the equilibrium point, and this is at a certain point, so I request you to please verify this problem by yourself. We have already found the equilibrium point. You can find the system matrix in the linear form, you can find the equation, and you can find the eigenvalues. Based on the polarity of the eigenvalues, we can comment. So, this is the structure in. So, let us stop here. We will continue from the next lecture on a new topic. A new topic will come up, and we will have a concept. Thank you.

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