Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 28

Stability of Nonlinear Systems

Hello everyone, in today's lecture we will be discussing the stability of the system based on the equilibrium point of the non-linear system, and also we will study how we can analyze the system around the equilibrium point. So, the first topic is the stability of the system. First, going to the mathematical part, let's define what stability is. So, a system is stable if, with a bounded input, the system yields a bounded output; then you can say the system is stable.

The system will be unstable if, for a bounded input, the output is unbounded. Without a bounded output, what does it mean? For example, we have a system here with a bounded input, the output, if it is also bounded, then we can say the system is stable. If the output is unbounded, the system is unstable. So, this is how we can define stability in a broad sense. That's another definition we can have: the system will be stable with zero or non-zero initial conditions if the resulting state trajectory

tends towards the origin. So, it means the system will be stable if the resulting output or trajectories tend to the equilibrium point. Here, we can say the origin is the equilibrium point, for example, x equals zero. As you have discussed in the last lecture, Even if the system does not have an equilibrium point at 0, we can make it 0 if we change the system or map the system to a different domain, and there we can make the system such that the equilibrium point will be 0, right. And if it is unstable, or we can write it as unstable. if the resulting trajectory tends to infinity. If the resulting output is diverging, for example, if the resulting output is this is a system

This is a system, for example, and this file code is for, and we have, we are simulating this system with some initial conditions, initial conditions, and if the resulting output, for example. Time axis, and if with time, if x(t) is going to infinity over time, then we can say the system is unstable. Now, let us assume the system $\dot{X} = f(X)$, this is, let us assume, an autonomous system. An autonomous system means the system does not depend on time. Let us assume X^* be the equilibrium point, equilibrium point, and x is the current state of the system, current state of the system.

If for all t, for all t, if x is always close to X^*, The system is stable. Now, let's formalize the notion of stability in terms of some theorem. So, there are different ways to define stability, so let's come up with some concept. It is called Lyapunov stability, so here. Let us, or you can write, there are different ways you can define stability, and one of them is Lyapunov stability, so Lyapunov stability. Let us consider the system. And let us assume X equal to 0 is an equilibrium point of this system, equilibrium point.

And also, let us assume this equilibrium point x equal to 0 at time t equal to t naught. Now, we have different definitions for this system, how it will be stable in different conditions. So, the first definition is, this is very important in non-linear control synthesis, the equilibrium point, equilibrium point X equal to 0 is. Is stable if for any epsilon greater than 0 and delta greater than 0, we are defining two different boundaries, epsilon and delta, can be found such that, such that the norm of the initial value of x, starting inside delta, implies that the value of X(t) will always be inside epsilon for all t greater than t_0. So, I will explain what this actually is, the definition.

So, let us assume we have this as the x-axis, X(t), and this is t. Let us also take it like this. We are defining two regions: one region is, for example, delta, and another region we are defining as epsilon. Similarly, minus delta, this is minus delta. And this is minus epsilon. Okay, now, if the system—if you look at the definition—if the initial value of x starts somewhere here, for example, it starts here and over time, it starts, for example, here. And over time, it will not cross the epsilon. It will always remain inside the upper region, which is less than epsilon.

The response of the system will be in this region. It will start from a region with a value less than delta and will not cross the epsilon region. Then, we can say this system is stable. So, we can write that this is stable. Okay.

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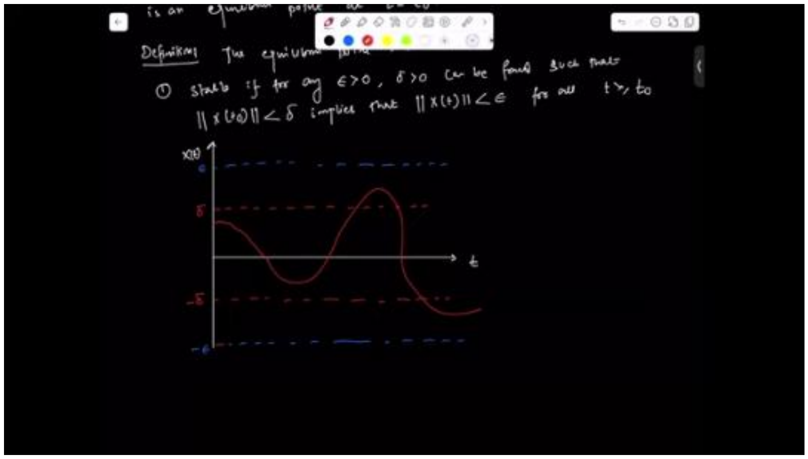

I hope it is clear. And if this condition, if the above condition, does not satisfy or fulfill, the system will be unstable. So, this is how we can define the stability of a system. Now, another point is asymptotic stability, asymptotic stability.

This is very very important. Here, if the system is stable and delta greater than zero can be chosen such that the initial value of x, x t naught, should start in the same region as the stable condition, but implies if it is asymptotic stability, it implies that limit t tends to infinity x of t goes to zero So, what does it mean? So, here the system should start in this condition, this condition, but over time the response can start like whenever it starts, but over time it will go to 0.

So, then what this means is as t tends to infinity, the system should go to 0. This is how we can define asymptotic stability. Now we have another condition for globally asymptotically stable. Globally asymptotically stable. So, here the condition is, if it is stable, of course, it will be stable, and limit t tends to infinity, X(t) goes to 0 for all, for all X, you know.

So, it can start anywhere in the space, it can start like this, but over time it will go to zero. So, this is what we should say. We can say this is basically this is it can start instead of this, it can start here also. I mean, outside all shapes are there. It does not matter where it is starting for all xt_0. So, this isasymptotically stable system. And this is now it is asymptotically, it is globally, it should be globally asymptotically. And this is this color is so this is how we can define the stability of the system, and also we can define how we can comment that the system is stable or not, or show whether the system globally is asymptotically stable or it is asymptotically stable also

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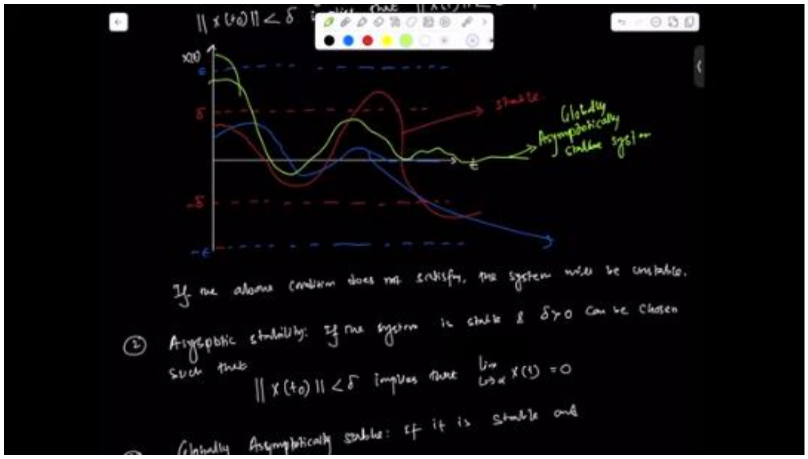

So, these are the important points while we will be discussing the Lyapunov stability based on the energy function; these are very important terms. Now, we will go to a very important part: how to analyze the system, a nonlinear system, through a trajectory, through the state trajectory concept. Before you move to that concept, first, in this nonlinear control concept, we will be doing, we will analyze the system; their system is a nonlinear system, a nonlinear system based on the two Lyapunov methods. So, Lyapunov's first method we'll explain in detail, the first method, and the second Lyapunov method. So, we'll go through this concept in detail with examples and Before we proceed to this part, first, let's analyze and classify the singular point.

So here, let us define some important topics: analysis and classification of singular points. These singular points can also be called equilibrium points, equilibrium points for a second-order system, a second-order system. So, this is a very, very important topic. So here, basically, in this part, we studied how we can define the stability of the system how we can define the equilibrium point, whether it is stable or not, based on these conditions. And now we will analyze the system, a nonlinear system, how to analyze whether it is stable or not. So, these are the tools we are going to discuss. So, let us assume we have a system

$$\dot{X} = f(X) \dots Eq(1)$$

and let us assume this is our autonomous system what one is system, and if you linearize the system, we can write the linearized system of this. I am not going into detail of it; we have done this thing multiple times in our first

But in the next few lectures, we will come up with the full concept, but for the time being, let me define the linearized system like this:

$$\delta\dot{X}(t) = \frac{\partial f}{\partial x}\bigg|_{Xe} \delta_{X.} \ldots Eq(2)$$

So, here X_e is the equilibrium point. equilibrium point, and $\frac{\partial f}{\partial x}$ is nothing but the Jacobian matrix. How we get this condition is basically we use the Taylor series expansion, Taylor series expansion, and we can get this linear form but we will come up with the full details of it maybe after a few lectures. So, for the time being, let us assume this is the linearized system we are getting from the nonlinear system.

And let us consider this, as we mentioned, this is a second-order system. We are going to consider a second-order system. So, for a second-order system, Let us define the linear system. What you are getting here is the second order. The second-order linear system is defined as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ldots Eq(3)$$

So, please don't follow the step first. We have the nonlinear system, and we are finding the equilibrium point. Around the equilibrium point, we are linearizing the system, and this linear system, if it is a second-order system, we are writing it in this form as equation three.

So here, this is the state matrix or sometimes called the system matrix. This is what we have done. Now, based on this system, we will come up with very important points. So, let me write based on the nature of the eigenvalues, the equilibrium point can be classified as first, node; second, focus;

or saddle point. So, this is the terminology we use in the nonlinear system because, in the nonlinear system, we are going to handle multiple equilibrium points in four bodies. So, how can we come up with these four different conditions on the western equilibrium point? Let's go step by step. We have the linearized model, and we are assuming the linear model in this form. Based on the eigenvalues of the system matrix, we can define what kind of nature the eigenvalues will be, and based on that, we can define the equilibrium point whether it is a node, focus, saddle point, or vortex. So, to find the eigenvalues, in the eigenvalues, so what is the process?

$$|\lambda I - A| = 0$$

$$\left|\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = 0$$

$$\lambda^2 - (a + d)\lambda + ad - bc = 0$$

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}$$

And this is basically nothing but the quadratic solution. So, now this is the quadratic equation, a second-order equation. Okay, so now it's very important if you remember we had some linear transformation concept, right? So now I'm not going into the detail of it, so I'm writing here straight away. Now, if we apply a similarity similarity transformation, we also have linear transmission as $X = MZ$. In the previous lectures in the modern control part, we had here T matrix, right.

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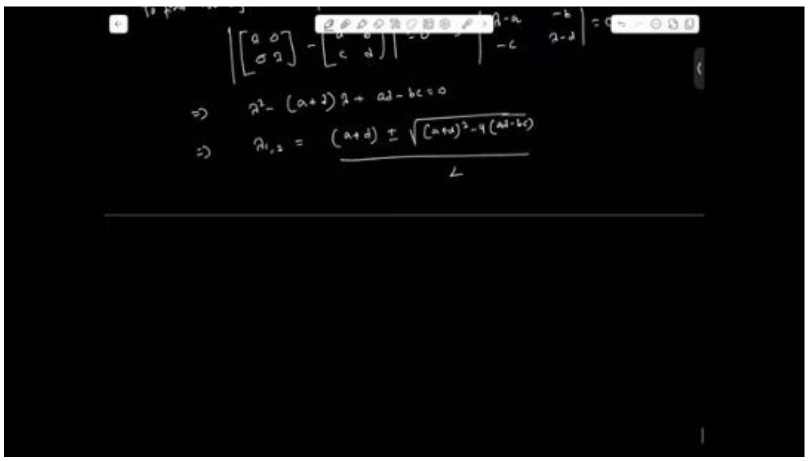

So, here we are assuming m to be the linear transformation matrix. So, here we are getting a different domain. From this, I am not going into the full steps. So, here we are getting a different system based on this transformation. So, we are writing

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

And from this, we are having, so this is the state z_1 and z_2. So here we are having two dynamics: one is the z_1 dynamics, which is

$$\dot{z}_1 = \lambda_1 z_1 \ldots Eq(4)$$

$$\dot{z}_2 = \lambda_2 z_2 \ldots Eq(5)$$

now, divide equation 5 by equation 4, we can write

$$\frac{dz_2}{z_2} = \frac{\lambda_2 dz_1}{\lambda_1 z_1}$$

So here, actually, we are analyzing the system with the equilibrium point, whether it's stable or not. So then we will start the control. Here, actually, we are doing all the analysis part, how we can analyze the non-linear system. So after we wind up these things, once you get the conclusion whether the system is stable or not, we can go for designing the control. And upon integration, if we apply integration from both sides, both sides get

$$\ln z_2 = \frac{\lambda_2}{\lambda_1} \ln z_1$$

$$z_2 = c z_1^{\frac{\lambda_2}{\lambda_1}} \ldots Eq(6)$$

So, here c is the constant. So, here if you notice carefully, we are coming to a very nice form here, we are having the state variable as well as eigenvalues in the system in this equation, right.

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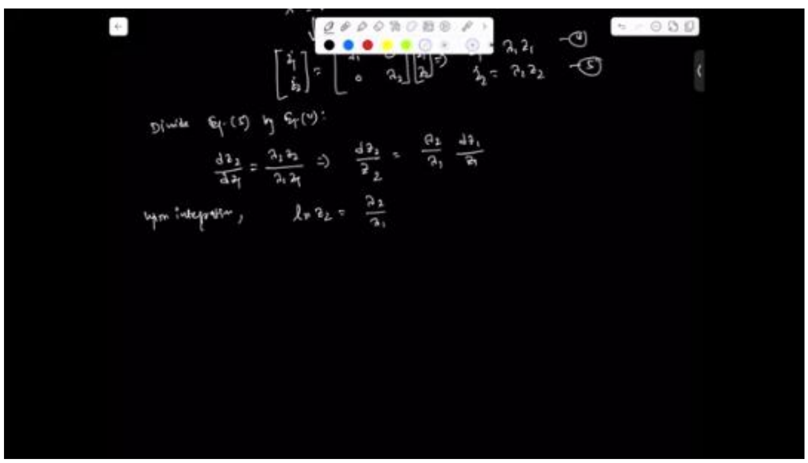

Okay. Or we can have another expression from these two equations, four and five, or we can write, since the systems are decoupled, we can solve individual equations. And we can come up with some closed ones, let me do it. From equation 4, we can write

$$z_1(t) = z_1(0)e^{\lambda_1 t} \ldots Eq(7)$$

and from equation 5, we can have

$$z_2(t) = z_2(0)e^{\lambda_2 t} \ldots Eq(8)$$

So, this is easy to solve because both equations are decoupled, 4 and 5 equations are decoupled, we can find this equation, right. So, from both equations, so here actually lambda 2 is not a function of t, this is simply t multiplied with lambda 2. Now, t from equation 7 and 8, we can write

$$z_2(t) = \frac{z_2(0)}{z_1(0)^{\frac{\lambda_2}{\lambda_1}}} z_1(t)^{\frac{\lambda_2}{\lambda_1}}$$

$$z_2(t) = c z_1(t)^{\frac{\lambda_2}{\lambda_1}} \dots Eq(9)$$

So, if you notice carefully, this term is actually constant because it has the initial conditions and the lambda, that is, eigenvalues. So, we can say this is the constant, and let us denote this as the same. So, we are getting the same expression as we had in equation 6. This is another way to get this expression. Now, I think we are So, we will define these conditions. We are now ready to explain about the node, focus, saddle point, and vortex based on this equation. But since today's time is up, we will continue in the next lecture from this part: how we can come up with these four different conditions and how we can comment on the nature of the equilibrium point because this is a very important part for studying the nonlinear system. Let's stop it here.

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