

# **Advanced Aircraft Control Systems With MATLAB / Simulink**

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**Lecture 19**

**Linear Quadratic Regulator**

So, here we are going to start the new topic, optimal control in modern control. Before we go to the main concept of the mathematical concept to derive the optimal control, let us work on the motivation of why we have to work on optimal control. So, let me write some points here. The first point is multi-input. Multi-input and multi-output system.

So, if you notice, in this case, we are handling  $\dot{X} = AX + BU$ . So,  $AX$  can be a vector, and  $U$  can also be a vector. So, there are multiple states involved in the system, as well as multiple control inputs. If you notice, we use the command 'place,' right? The place command, and by using the place command, we can place the poles to the desired location, the desired location, right? But using this concept, this place concept, the way to place the poles in the desired location, I cannot fulfill all the desired parameters, cannot fulfill all the desired parameters.

Of the system objectives, this is very, very important because we are here, actually, a limited number of them could be found on the closed-loop pole locations. We cannot capture all the design parameters in the control synthesis. The second problem is to determine, determine. The desired poles are also very challenging. Which poles will give us the desired results? It is also very challenging, challenging to find, to find the optimal, optimal poles. Which can fulfill the performance objectives? So, this is also challenging: how to find the poles which can help us to come up with the optimal results. So, using the concept that we have designed in the state feedback control, it is quite challenging, which can provide the optimal results. Another problem is, one condition can be, if there are many possibilities, if there are, how to find the.

Solutions on how to find, so all these challenges can be achieved if you have the optimal control. Like this, the above challenges can be achieved by using the optimal technique. So let us see how we can design optimal control. So, optimal control basically we are going to choose some objective function which will fulfill the design parameters and different vision parameters in the system which can take into picture and also which can

find the best solution For example, these are the desired values of the system. And the response of the actual system, for example, is something like this. So, if you notice, after some time, the system goes to settle down after some time.

And before this system goes to settle down, it goes through a number of overshoots and undershoots, right? So, before this goes to settle down, this part, for example, is the time it goes to settle. And let's assume this is  $t_s$  Ts, the time it takes to settle down  $t_s$ , and the time till  $t_s$  we call in control system transient response. And after it goes to, after it settles down, the future time after  $t_s$  is called steady state, steady state condition, right? But if you notice the system till it goes to settle down there is some transient energy in the system and which is due to the overshoot and undershoot in the system. So if you want to reduce this overshoot and undershoot and also if you want to achieve your system to reach faster to the steady state condition, then we need to apply some control So, it means there are two important components: one is transient energy, transient energy, and another component is control energy, control energy. So, we need to, using these two conditions, we need to find the objective function.

That objective function will contain one component, transient energy, and another component, fundamental energy. So, we have to balance between them so that we can get an optimal result. And that optimal, once it goes to the optimal condition, we will come up with the best possible control. And that control is basically nothing but the control gain matrix.

And this objective function we defined down the cost function which is defined as

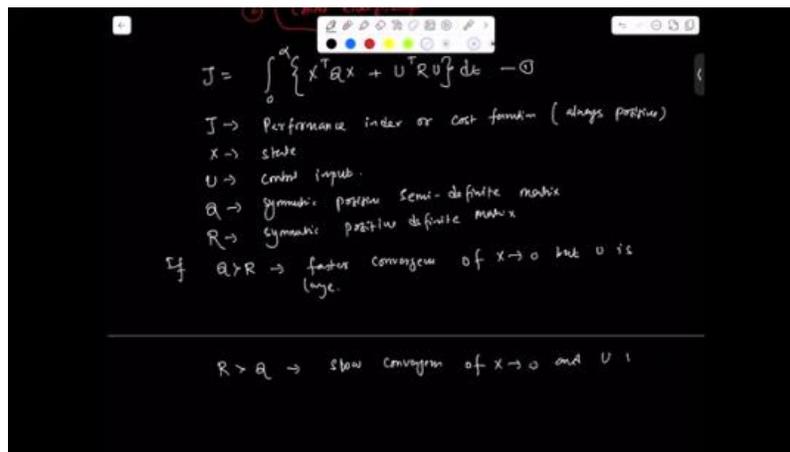
$$J = \int_0^{\infty} \{X^T Q X + U^T R U\} dt \dots Eq(1)$$

From this equation, we can see that the control energy and transient energy with respect to time have been defined. Therefore, the objective function of the optimal control problem must be time integral so that's why we have taken time integral here must be time integral here of the sum of the transient energy and this is the control energy so now our main aim is how we can minimize this function while and minimize it, and we can reach the desired position very quickly. Now let us define what are the parameters in equation 1 so here J is actually the performance index or sometimes called the cost function is positive and x is the state vector state and u is the control input and Q is the symmetric semi-definite matrix. Semi-definite matrix means the determinant of the

matrix can be 0 or it has been 0, it can be negative. And R is a symmetric positive definite matrix.

So here if you notice here we have two important conditions for this equation one if Q is greater than R we can get faster convergence of X which is supposed to go to zero we need to apply more control effort so it means U is large and if R greater than Q then convergence can be slower slow convergence of X tends to 0 and u is small so it is obvious if you apply more control input if you put more energy in the system you can get to the desired value first and if you apply less energy you can reach the desired location or position not very fast now we'll come up with the mathematical formulation this process is sometimes called the linear quadratic regulator problem the above process of designing control is called the linear quadratic regulator problem so what does linear mean? Linear means we are working on the linear system quadratic basically here the cost is these are functions of quadratic form because these are quadratic forms that's why this quadratic and regulator basically we are designing the controller right a regulator

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That is why it is called the linear quadratic regulator problem. Now, we are going to introduce. So, now we are going to do the mathematical problem before that. So, for that, let us introduce a new matrix. Let us

Introduce a symmetric matrix, symmetric matrix P, which satisfies the condition

$$P^T = P$$

P transpose equal to P. And if you modify equation 1, now if you modify and modifying Equation 1 as

$$J = X_0^T P X_0 - X_0^T P X_0 + \int_0^{\infty} \{X^T Q X + U^T R U\} dt$$

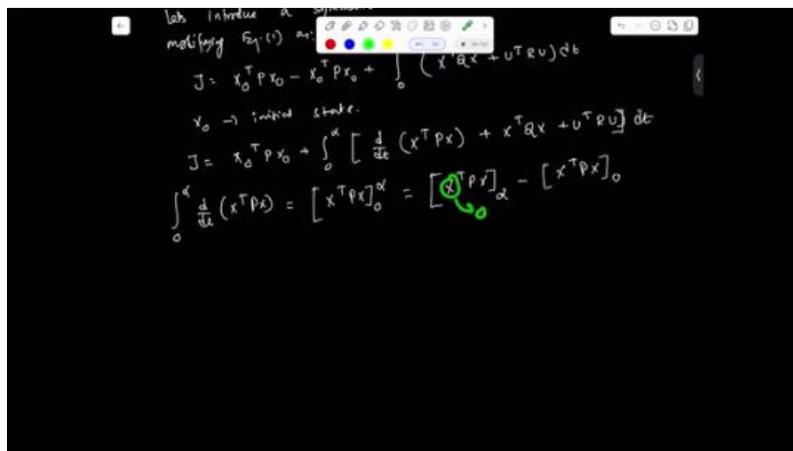
Z equal to X naught transpose B X naught plus, sorry, minus X naught transpose So, here we have introduced this term plus minus. So, here basically  $X_0$  is the initial state. So, if we introduce this term, this our J, first function say,

Further, J can be written as

$$J = X_0^T P X_0 + \int_0^{\infty} \left[ \frac{d}{dt} (X^T P X) + X^T Q X + U^T R U \right] dt \dots Eq(2)$$

$$\int_0^{\infty} \left[ \frac{d}{dt} (X^T P X) \right] dt = -X_0^T P X_0$$

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So, I hope it is clear now. Let's find the derivative

$$\frac{d}{dt} (X^T P X) = (A X + B U)^T P X + X^T P (A X + B U) \dots Eq(3)$$

Let's go back to again function j,

$$J = X_0^T P X_0 + \int_0^{\infty} [(A X + B U)^T P X + X^T P (A X + B U) + X^T Q X + U^T R U] dt$$

$$(A X + B U)^T P X + X^T P (A X + B U) = X^T [A^T P + P A] + U^T B^T P X + X^T P B U \dots Eq(4)$$

And now, we can use this term in equation 2. We can substitute So, now the first function

$$J = X_0^T P X_0 + \int_0^{\infty} [X^T (A^T P + P A + Q) X + U^T R U + X^T P B U + U^T B^T P X] dt$$

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$$\begin{aligned} \frac{d}{dt} (x^T P x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax + Bu)^T P x + x^T P (Ax + Bu) \quad \text{--- (3)} \\ J &= x_0^T P x_0 + \int_0^{\infty} [(Ax + Bu)^T P x + x^T P (Ax + Bu) + x^T Q x + U^T R U] dt \\ (Ax + Bu)^T P x + x^T P (Ax + Bu) &= (x^T A^T + U^T B^T) P x + x^T P A x + x^T P B U \end{aligned}$$

So, here we will further simplify this expression. So, let us work on the last three terms. Let us work. So, let us work on how we can further simplify this expression, OK?

Let us define this term  $U^T R U + X^T P B U + U^T B^T P X$  as term 1 and substitute. So, term 1 we can write as:

$$\begin{aligned} &U^T R U + X^T P B U + U^T B^T P X \\ &= (U + R^{-1} B^T P X)^T R (U + R^{-1} B^T P X) - X^T P B R^{-1} B^T P X \dots \text{Eq(6)} \end{aligned}$$

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$$\begin{aligned} &(Ax + Bu)^T P x + x^T P (Ax + Bu) \\ &= x^T [A^T P + P A] + U^T B^T P x + x^T P B U \quad \text{--- (4)} \\ \text{The cost function:} \\ J &= x_0^T P x_0 + \int_0^{\infty} [x^T (A^T P + P A + Q) x + \underbrace{U^T R U + X^T P B U + U^T B^T P X}_{\text{Term 1}}] dt \end{aligned}$$

So, let us substitute 6 into equation 5. So, we are getting

$$J = X_0^T P X_0 + \int_0^{\infty} [X^T (A^T P + P A + Q - P B R^{-1} B^T P) X + (U + R^{-1} B^T P X)^T (R U + R^{-1} B^T P X)] dt$$

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Handwritten derivation on a blackboard background:

$$\begin{aligned} \text{Term 1} &\rightarrow U^T R U + U^T B^T P X + X^T P B U \\ &= U^T R U + U^T B^T P X + X^T P B U + X^T P B R^{-1} B^T P X - X^T P B R^{-1} B^T P X \\ &= (U^T + X^T P B R^{-1}) (R U + B^T P X) - X^T P B R^{-1} B^T P X \\ &= (U + R^{-1} B^T P X)^T R (U + R^{-1} B^T P X) - X^T P B R^{-1} B^T P X \quad \text{--- (6)} \end{aligned}$$

(We substitute (6) in Eq (5):

$$J = X_0^T P X_0 + \int_0^{\infty} [X^T (A^T P + P A + Q) X + (U + R^{-1} B^T P X)^T R (U + R^{-1} B^T P X) - X^T P B R^{-1} B^T P X] dt$$

if you notice there are two important terms in this equation J. From the last equation, last term, please note carefully, from the last term of this equation, this is the last term, right? This term, how can we make it zero? If you make this equal to zero, right? From the last term, we can easily write, so here, if you select

$$U = -R^{-1} B^T P X \dots \text{Eq(7)}$$

then the whole term goes to 0 right and all you know from our last lectures in the state feedback control, we already know

$$U = -K X \dots \text{Eq(8)}$$

So, now if you compare both these equations 7 and 8. Comparing equations 7 and 8, we have

$$K = R^{-1} B^T P \dots \text{Eq(9)}$$

We have found a very, very important relation for the control K. So this is a very, very important relation.

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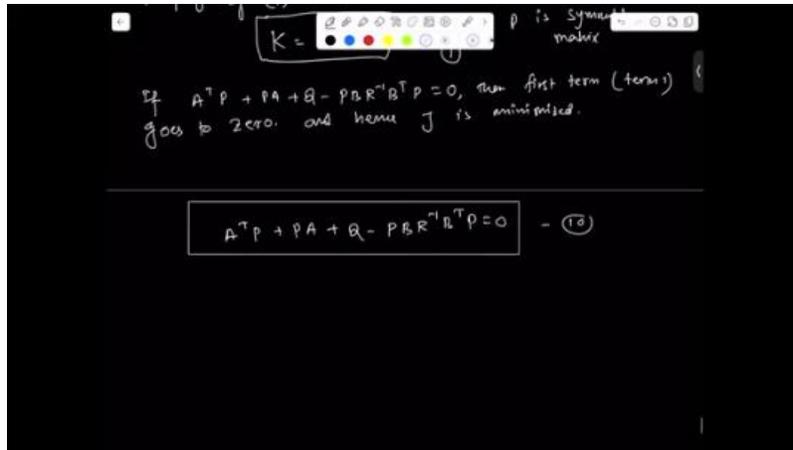
So here, if you notice in this equation, we are guessing some value R. P is given to us, so R is already a different matrix, and P is some semi-definite matrix we have to choose, right? So here we can write that P is a symmetric matrix, but here if you notice, we don't have we need to find the P matrix. We don't have the expression for P, so this P we can find if you solve this expression again because here what you are given is Q. Q we choose as a matrix, so here you already choose a symmetric positive definite matrix or matrix, and also R also equation A9, the only unknown is P, so P now we have to find. This P you can find from this term, term 1, if you denote this as term 1, this is So from term one, we can write.

So what is the condition again? We can assume if we can make the inside part, the inside, if we can make, if we consider, to be 0, then we can say J can be minimized because we already came up with this condition by assuming this condition to 0. So term 2 can be minimized if you consider this again. If you consider this term equal to 0, then we can indirectly say that J can be minimized because both the terms inside the bracket go to 0. So now we can write if

$$A^T P + P A + Q - P B R^{-1} B^T P = 0 \dots Eq(10)$$

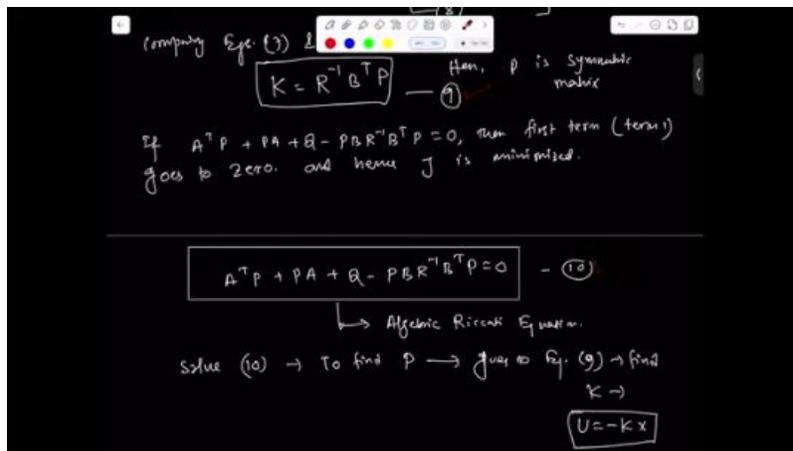
then the first term goes to zero, and hence J is minimized. That is our main goal: how to minimize J minimized. So, by using equation 9 and 10, we can design optimal control or linear quadratic regulator in your problem. This equation is sometimes also called the algebraic equality equation.

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So, if we solve equation 10 to find p, and this p goes to equation 9 to find a, and this k goes to  $U = -KX$ . So, this is how we can design optimal control. So, if you notice, we started with J and we had some mathematical manipulation, and we have two conditions: condition 9 and 10. These are the optimal conditions which can make the cost function J to be minimized. So, that's why we call it an optimal problem because we are minimizing some function to get the optimal result of a problem.

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This is a very, very powerful technique for designing the control algorithm for a dynamical system. And we'll take an example because today's time is short. We'll take an example of how we can implement this LQR control. This is also called LQR. So, we will see how we can use this concept for any problem we come up with in the next lecture.

Thank you.