Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 14

Example to demonstrate use of Ackermann's formula

In today's lecture, we will be taking an example and seeing how we can validate the concept we have discussed in the last lecture. So, let us set the example. Consider an inverted pendulum. With the following state space equation. Here we are given the A matrix of the system.

$$A' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.78 & 0 & 0 & 0 \\ -0.95 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

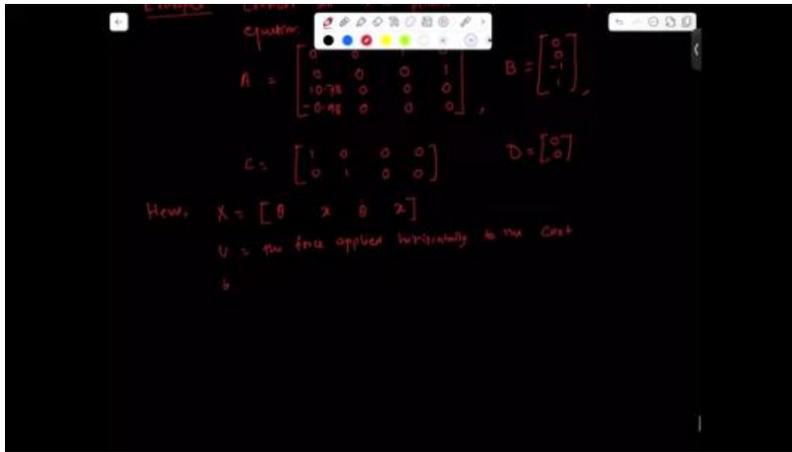$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So there is no feedthrough controller in the system. So here, the state vector. X is the state vector:

$$X = \begin{bmatrix} \theta & x & \dot{\theta} & \dot{x} \end{bmatrix}$$

and u is the control input: the force applied horizontally. And theta is the angular position of the pendulum, and x is the horizontal position of the cart.

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And $\dot{\theta}$, $\dot{x}$ are the respective velocities. The first question is to design a full-state feedback regulator. Okay, the second question is to design an observer using only one of the states. So, let us solve this problem. We will be using the concepts we have discussed in the last lecture.

From the given system matrices, you can find the system equation in LTI form. This part we have already done multiple times. So now we'll start with the solution of the problem. So here, first let's design the state feedback.

So here we'll be using the concept of the Ackermann formula. Before that, we need to go through the basic steps for designing control. The first step is we need to find the P matrix. P matrix, P is basically used to test the controllability of the system. So we can directly test CTRB, A comma B,

and then we will find the rank of P. If you enter, we will get r equal to 4. So if r equal to 4, it is quite clear that such a system is controllable. Hence, the system is controllable, so we can design the state feedback control. Also, let's check whether the system is stable or not. For that, we can use the DAMP command in MATLAB to check system stability. For that, we can use this command: omega m zeta, omega n is the natural frequency, zeta is the damping ratio, and P is a vector which contains the poles of the system, determining whether the system is stable or not. So, if you enter this command in MATLAB, we have the values omega, zeta, and poles. So here we are getting

$$
\begin{array}{ccc}
\omega_n & \xi & P \\
0 & -1 & 0 \\
0 & -1 & 0 \\
3.2833 & -1 & 3.2833 \\
3.2833 & -1 & -3.2833
\end{array}
$$

If you notice in the pole location, one of the poles lies in the right-hand side, so this is quite obvious that the system is not stable, right? So, the system since since $s = 3.2833$ s, so here, basically, one of the poles lies in the right-hand side of the s-plane, and the system is not stable. So, we need to design the state feedback control, which will help us to shift the unstable pole to a stable regime. So, for that, let us consider, let us select a closed loop poles at

$$S = -1 \pm i$$

$$S = -5 \pm 5i$$

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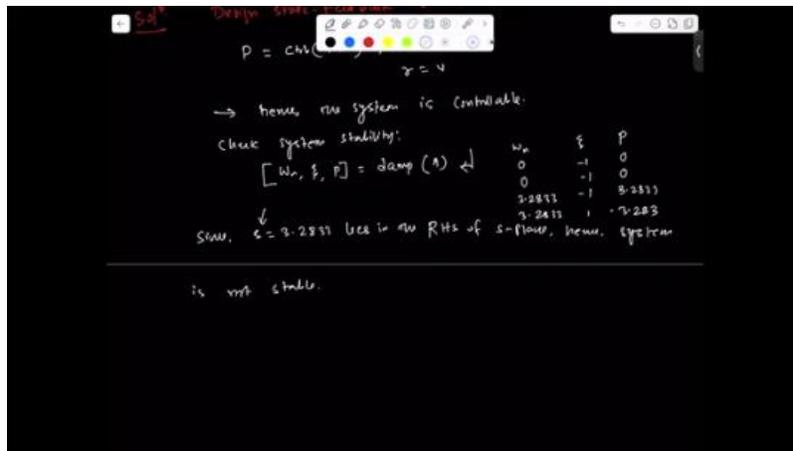
Now, we can use the command a, so we can use, we can use, we can use For that, we can, we have already found the formula

$$K = (\alpha - a)P'P^{-1}$$

So here, if you notice, we have to find these parameters: alpha, a, b; we have to find. So, for that, we can first, so here one thing: alpha are the coefficients, alpha basically the coefficient of the characteristic equation, the characteristic equation, and a is the, here a is the coefficient of the, our characteristic equation of the system and it is, right.

So here, if you write alpha, it is actually nothing but alpha that you have done in the last lecture: alpha is

$$\alpha = \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & \alpha_0 \end{bmatrix}$$

$$a = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix}$$

So, this is basically the closed-loop characteristic polynomial we are getting this, this is the coefficient of ss, and this is basically the plant characteristic polynomial, the characteristic equation of the polynomial, right? Since here in, we found four, we found four because it is fully controllable, and now let us find the plant characteristic equation, the plant's characteristic equation. So, we can find this

$$|SI - A| = 0$$

and if you find it difficult to solve the mat characteristic equation, you can use the MATLAB command So, here we can write A equal to poly.

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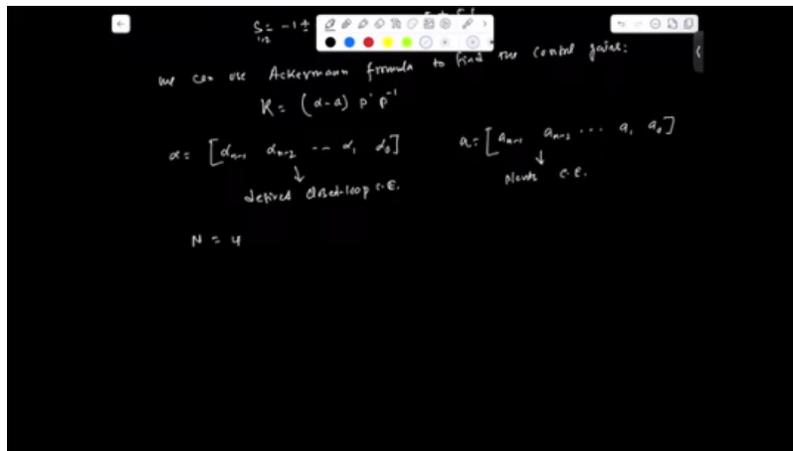
So, if you enter the MATLAB command, you will get

$$a = 1 \quad 0 \quad -10.78 \quad 0 \quad 0$$

And the characteristic equation from these roots you can find

$$S^4 + 0 * S^3 - 10.78 * S^2 + 0 * S + 0 = 0$$

$$S^4 - 10.78 * S^2 = 0$$

That's it. So here we can write, if you compare this with

$$S^4 + a_3 * S^3 + a_2 * S^2 + a_1 * S + a_0 = 0$$

$$a_3 = 0, a_2 = -10.78, a_1 = 0, a_0 = 0$$

So, using these terms, we can form the P' matrix.

$$P' = \begin{bmatrix} 1 & -a_3 & -a_2 & -a_1 \\ 0 & 1 & -a_3 & -a_2 \\ 0 & 0 & 1 & -a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 10.78 & 0 \\ 0 & 1 & 0 & 10.78 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, also, we need to find the p dash inverse because in the gain matrix, we have this expression. We have found alpha. Sorry, we have found a p dash. We have found now we have to find P'. This is this should be P'. Similarly, we need to find alpha. For this, we have the desired roots given to us.

$$V = [-1 + i \quad -1 + i \quad -5 + i \quad -5 - i]$$

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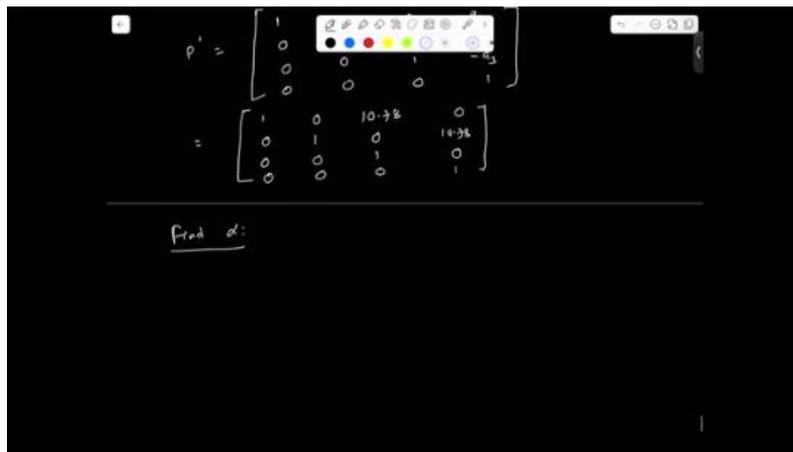
As polynomial coefficients separately, alpha are known to be

$$\alpha = 1 \quad 12 \quad 72 \quad 120 \quad 100$$

So, from this, we can find the polynomial of alpha. So, we can write

$$S^4 + \alpha_3 * S^3 + \alpha_2 * S^2 + \alpha_1 * S + \alpha_0 = 0$$

And this is basically we can write

$$|SI - A_{CL}| = S^4 + 12 * S^3 + 72 * S^2 + 120 * S + 100$$

$$\alpha = [12 \quad 72 \quad 120 \quad 100]$$

$$a = [0 \quad -10.78 \quad 0 \quad 0]$$

Now, we can find the K

$$K = (\alpha - a)P'P^{-1}$$

$$K = [-92.98 \quad -10.20 \quad -24.24 \quad -12.24]$$

and this is the control matrix gain matrix which is going to find the state feedback control, right? This is, if you write this command, if you write this line in MATLAB, you can find this K matrix. Or, even if you find difficulty, simply also, there is the simplest way you can find also, K equal to, you can write, you can write K equal to ACKER, A, B, and V.

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If you write this command in MATLAB and if you enter it, you'll get the same

$$K = [-92.98 \quad -10.20 \quad -24.24 \quad -12.24]$$

So instead of going through all these steps, you just simply write this command in MATLAB: A, B, and P. So, A is the system matrix, B is the control matrix, and B is your desired poles of the system to be used to design the control. So, this is how we can design the state feedback control in MATLAB. Now, we will come to the second part of the question, B: we need to find the observer, observer design, right? We need to consider,

let us consider, let us consider that the angular position of the pendulum is available for measurement.

Okay. Then, if it is available for measurement, then we can find.

$$Y = CX = [1 \quad 0 \quad 0 \quad 0]X$$

So here we can write power $Y = \theta$. That's it. So we are assuming this. Now, based on this assumption, let's check whether the system is observable or not. So check observability. So we can use the command. So we need to find the N matrix, right? So the N matrix you can find O B S V (a,c) and if you find our rank of n we get r equal to 2 so it means if you consider our c matrix if you consider c matrix in the structure my system is not observable so plant is not observable this is you have to play in this design the observer you have to think how can find how can choose the symmetry that the system is observable so this is a very important part for designing observer so now we have to again redefine the semantics so in the second let's have a second try, let's check let's assume let's assume output output measurement to be first position first position only So in this case, we can say

$$Y = CX = [0 \quad 1 \quad 0 \quad 0]X$$

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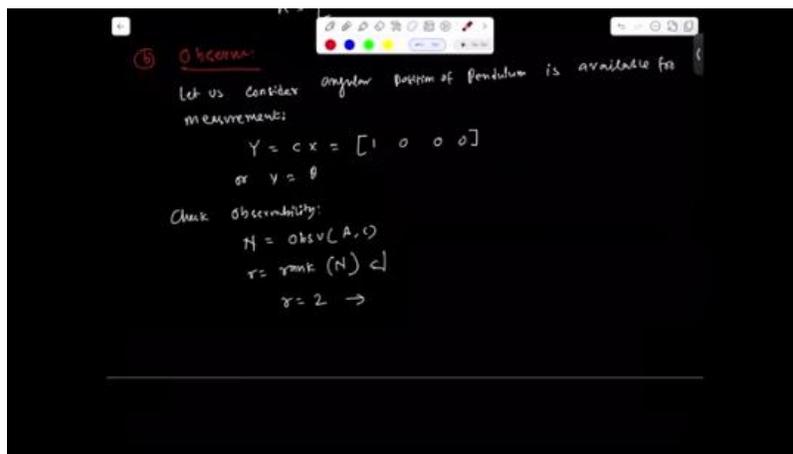
Now again, let's check the rank of the observability test matrix. So let's find n if you write ob as v comma c, and if you find r equal to the rank of n, and if you enter it in the command window, we can get r equal to 4. Just now, the system is observable. It is a full-rank system and is observable. So now, then we can proceed the same way as we have done before. Now design the observer for that. First, we need to find the observer gain

matrix, right? So for that, we need to, let's assume, let's consider, let's consider the observer. Observed poles at s equal to, so let us consider the observed desired poles, desired poles at

$$S = -10 \pm 10i$$

$$S = -20 \pm 20i$$

So If you remember, we considered in the previous lecture that the observable poles should be more negative compared to the regulator poles. Right, so if you notice, the regulator poles are much lower than these values in magnitude.

So now we can form the V matrix here.

$$V = [-10 - 10i \quad -10 + 10i \quad -20 - 20i \quad -20 + 20i]$$

and if you write P polynomial P V if we write, we will get the entire p. We are getting
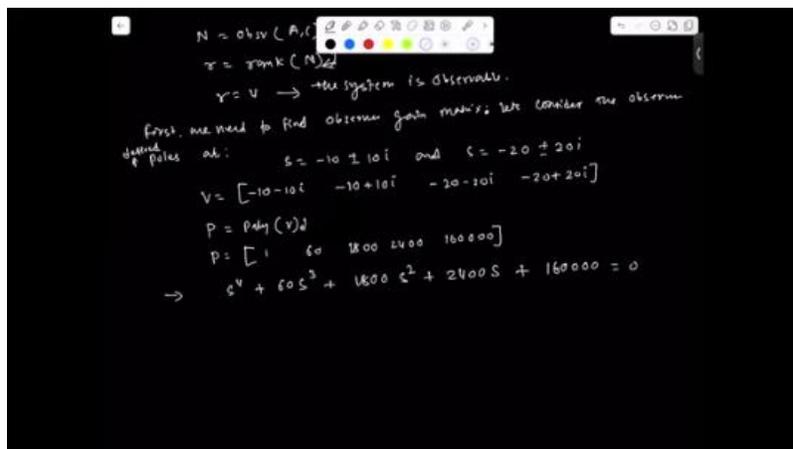
$$P = [1 \quad 60 \quad 1800 \quad 2400 \quad 160000]$$

And from this, we can write the desired characteristic equation

$$S^4 + 60 * S^3 + 1800 * S^2 + 2400 * S + 160000 = 0$$

So here it is nothing but β

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$$\beta = [60 \quad 1800 \quad 2400 \quad 160000]$$

$$a = [0 \quad -10.78 \quad 0 \quad 0]$$

So now we can use, we can find L, the observer, the observer gain matrix.

$$L = [(\beta - a)P'N^{-1}]^T$$

$$L = [-25149 \quad 60 \quad -183183 \quad 1810]$$

So, this is the observer gain matrix. This is numerically you can do. So now if you, so using this gain matrix, we can find the observer. Observer's augmented system or systematics, you can write. Let's write system matrix, so this is

$$A_0 = A - LC$$

And this is basically to remember we had E dot naught equal to

$$\dot{E}_0 = A_0 E_0$$

the eigenvalues of A_0 should be negative. So, that E_0 goes to 0 as t tends to infinity. This is the main motivation for designing observer, A naught between the actual state and estimated state should be 0. So now if you find the eigenvalues, sorry, eigenvalues of A naught, so let us define it equal to this. If you enter it, we get the eigenvalues

$$-20 \pm 20i$$

$$-10 \pm 10i$$

So, if you notice these eigenvalues of this A naught matrix, all lie in the left-hand plane or lie in the left-hand plane, one may be here minus 10 plus may be here is 20. So, one is here, one is here. Eigenvalues location and all eigenvalues are on the left-hand side, so the system is stable. You know it is going to zero, so the system is stable. All states are available in the measurement, and all states are estimated to design the state feedback control. Therefore, you can easily find the control algorithm for the regulator. Now we have prepared the code for this system. So these are the codes you can use. So for this particular example, here is the controller design part, the control design part, and this is the part for observer design. So you can use this code for your reference.

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And you can validate whatever mathematical things you have done in this whole lecture. In the same way, you can proceed with the observer and controller design for other systems, such as aircraft or any other aerospace systems. You just have to change the matrices, that's it. So, let's stop here, and we'll continue from the next lecture. Thank you.