Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 13

Ackermann's Formula

Here, we are going to start the main concept: how we can design the controller. So, we'll be using the Ackermann formula for pole placement controller design and also the observer design. So, here, let's assume we have the dynamics that we write for a system. For a SISO system, The form, let's assume

$$y^n + a_{n-1}y^{n-1} + \cdots + a_1\dot{y} + a_0 y = b_0 u \ldots Eq(1)$$

So, this is the system where we are having single input and single output. So here, we have the input, and we have y is the only state available in the system. Now, for the system, if A and B, the system and control matrices are written as

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \ldots & 0 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} b_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ldots Eq(2)$$

Where $a_0 \ldots a_{n-1}$ are the coefficients of the plant's characteristic polynomial or characteristic equation. Okay. Where we can find the characteristic equation

$$|SI - A| = S^n + a_{n-1}S^{n-1} + \cdots + a_1 S + a_0 \ldots Eq(3)$$

So the coefficients of this correct equation are being used for finding the A matrix. Okay. Go to the coefficients of equation one, the system equation. And if you can write the matrix A and B for this particular system, for this particular system, then we can say the system is in controller companion form, so then we say that the plant is in controller companion form. If you can do this structure, if you can find the system matrices in this structure, it will actually be helpful for us in designing the control. If you would like to know more about this, you can find the reference book Friedland. This is already mentioned in the course handout in the outside Friedland book. You can go to the modern control part; there, detailed things are done. Now, as you have done the state feedback control, the gain matrix for state feedback in matrix, we have

$$K = [K_1 \quad K_2 \quad \dots \quad K_n]$$

because we are controlling n number of states. The dimension of this equation is n, so now

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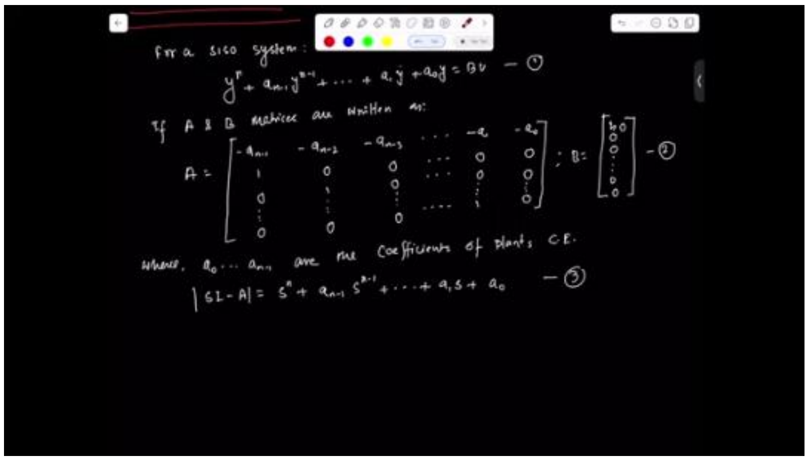

Based on this, based on this, uh, control gain matrix, we can find the augmented matrix, which is the augmented matrix already done. The augmented matrix we can write as $A_{CL} = A - BK$ right? And if you find the characteristic equation of the augmented system, the characteristic equation of the augmented matrix AM, we can write it as

$$|SI - A + BK| = S^n + \alpha_{n-1}S^{n-1} + \dots + \alpha_1 S + \alpha_0 \dots Eq(4)$$

Now also we can write that this is the characteristic equation, right? This is the characteristic equation in this equation form. Now A - BK also we can find in matrix form, so A_CL = A - BK also we can write.

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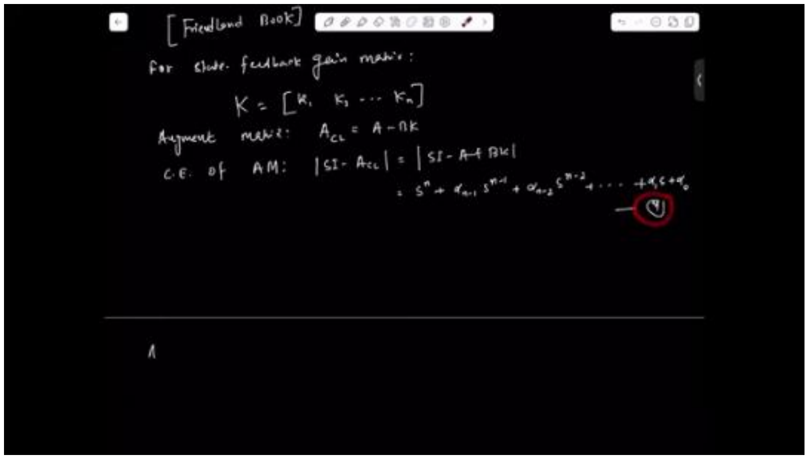

$$A_{CL} = \begin{bmatrix} -a_{n-1} - K_1 & -a_{n-2} - K_2 & \cdots & -a_0 - K_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \ldots 0 \end{bmatrix} \ldots Eq(5)$$

closed-loop system is also in companion form. So this system is also in companion form, this A - BK. So let us write this as equation number 5.

Now if both are in companion form, we can compare equation 5 and equation 4, and we can come up with the Comparing, or we can write from equation 3,4, the coefficients, the coefficient of Closed-loop equation of 3,4 here, this is equation 4 and equation 5. Now, the coefficient of the closed-loop current characteristic equation must be equal to Equal to equation 4, right? So in this condition, we can write

$$K_1 = \alpha_{n-1} - a_{n-1} \qquad K_2 = \alpha_{n-2} - a_{n-2} \qquad K_n = \alpha_0 - a_0$$

and if you write in vector form

$$K = \alpha - a \ldots Eq(6)$$

So here we can write

$$\alpha = [\alpha_{n-1} \quad \alpha_{n-2} \quad \cdots \quad \alpha_0] \qquad a = [a_{n-1} \quad a_{n-2} \quad \cdots \quad a_0]$$

okay now um if you assume state space representation is not in uh okay these are the conditions we are having if the

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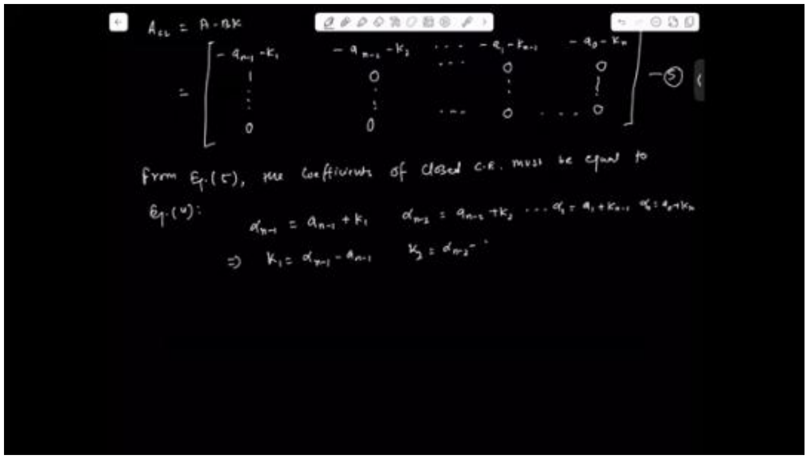

will handle are not in companion format so for that case we already have done the state transformation the way basically helps us to convert any system into controller companion form so if the system in the system it's not Is it not companion? Companion. Controllable. Controllable form.

You can use. This. Transformation. already you have done how to convert the system into a diagonalized form so here we can write $X = TZ$ so X is the original state and Z is the another state in another domain and T actually turns from one domain to another domain so we get from this the original system is our

$$\dot{X} = AX + BU$$

$$Y = CX$$

so since we are designing the controller so here this equation is very important so from this equation you can substitute this value in this equation we get which we already have done in our previous lecture

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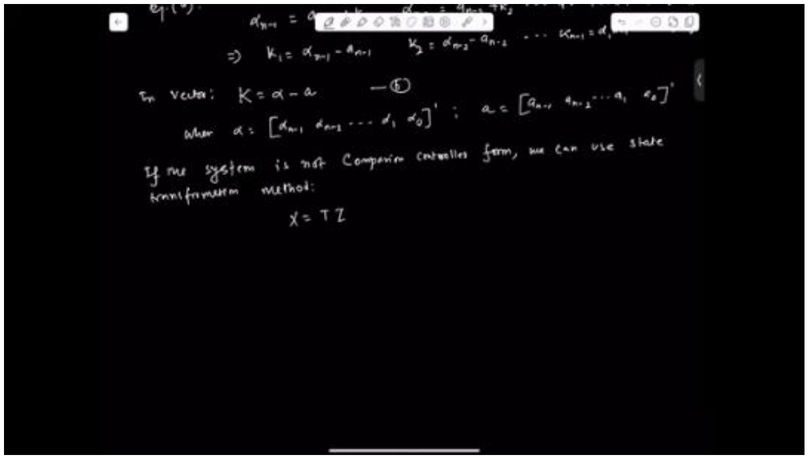

$$A' = T^{-1}AT \Rightarrow A = TA'T^{-1}$$

$$B' = T^{-1}B \Rightarrow B = TB'$$

Okay, so now for this transform system, you can also design the controller for the regulator. This is a regulator problem. We are assuming X to the 0 here. It is your statement, so it is a regulator problem. For regulator control, you can write

$$U = -KX = -KTZ \ldots Eq(6)$$

Now here, we can write since KT is the regulator gain matrix. When the plant is in controller companion form, so now what I'll do is, if you compare the controller what you have done here and what you are having here, so if you compare both the. Stuff, we can write.

From equations 5 and 6, we can write

$$K = (\alpha - a)T^{-1}$$

So here, we need to know T inverse, which will transform the plant to a favorable form. This is already known.

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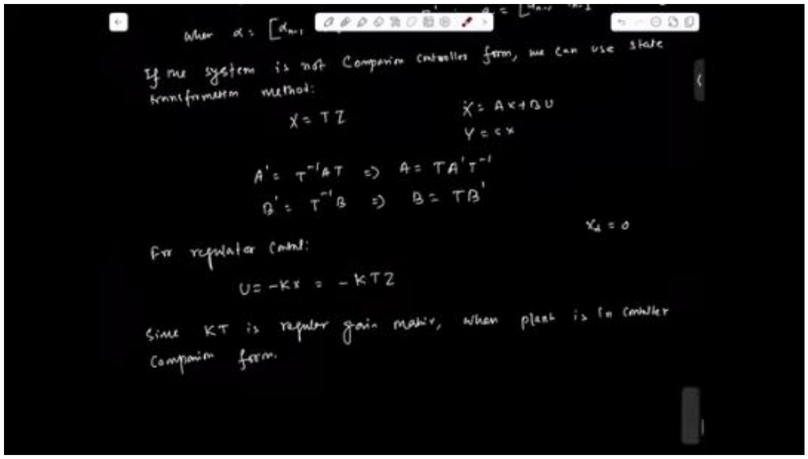

Now, we will take the controllability test matrix and, using which, we will come up with the final Ackermann formula to find the controller gain of the system. So, as you have done the controller test matrix. Controllability test matrix, we have done already, so we had

$$P = [B \quad AB \quad A^2B \ \ldots \ A^{n-1}B]$$

$$= TP'$$

Where

$$P' = [B' \quad A'B' \quad A'^2B' \ \ldots \ A'^{n-1}B']$$

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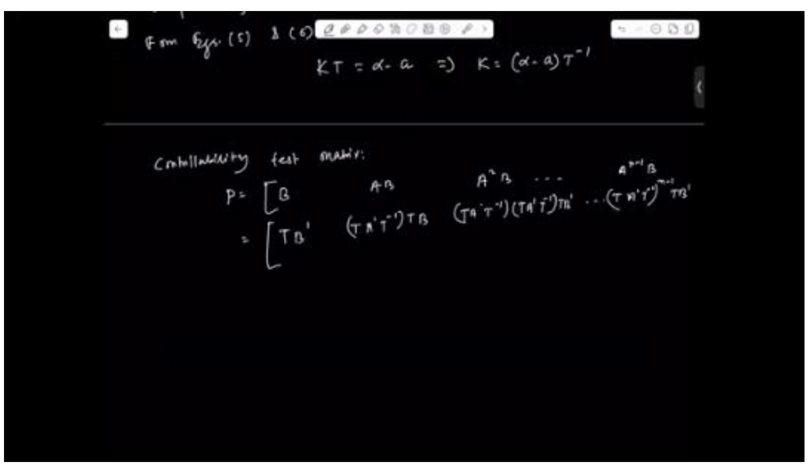

Now pre-multiply, we are trying to get the simplified form, pre-multiply by T^{-1} on both sides We are having

$$T^{-1}P = P'$$

and again if we want to have post multiply by P^{-1} on both sides

$$T^{-1} = P'P^{-1}$$

now also we need to just important note here P' Actually, upper triangular matrix, so upper triangular matrix means the one of the I mean the The elements below the diagonal elements, the diagonal lines are zero So here we can find the P' matrix For this is a very good system in the aircraft or any other system, and how we can do all the steps for a in the example will be showing in the next lecture, maybe you find difficulty in this mathematics, but as you will be solving them problem, it will be easy for you, so P dash obtained as

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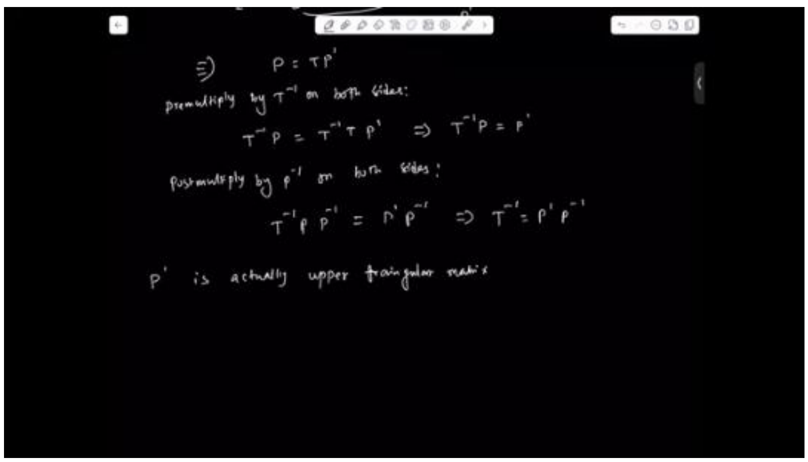

$$P' = \begin{bmatrix} 1 & -a_{n-1} & \dots & -a_1 \\ 0 & 1 & \dots & -a_2 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & \dots 1 \end{bmatrix}$$

so now if you substitute this term t inverse in our equation where we have written yeah in this equation so in place of t inverse we can write p dash p inverse so the expression the expression

$$T^{-1} = P'P^{-1}$$

and we can write

$$K = (\alpha - a)T^{-1}$$

and this is called the Ackermann pole placement formula for finding the controller matrix k this is called a very powerful formula called Ackermann, now if you want to know the same thing for the observer design how can we the second part we are going to start here in this course second part how can we find the observer design using a common formula so the pole placement can be carried out in a similar manner to what we have done in the pole placement design for the controller so here let us write for plant equation which we write

$$|SI - A| = S^n + a_{n-1}S^{n-1} + \cdots + a_1S + a_0$$

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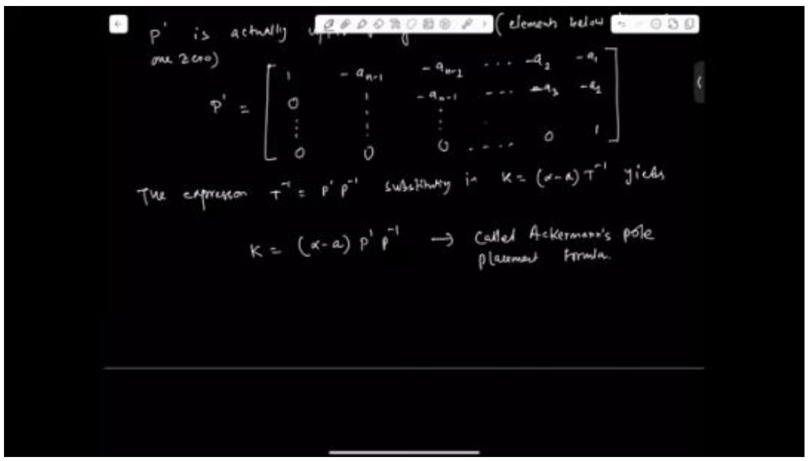

now if you see here we are finding the closed loop the closed loop characteristic equation right here and also we are finding the matrix $A - BK$ similarly we can do for the observer because in control design we are using k but in the observer design we are using l if you remember the last lecture we are designing l for the observer design the same structure so here we can write we can find observer in matrix L which places observer poles such that observers characteristic equation is

$$|SI - A_0| = S^n + \beta_{n-1}S^{n-1} + \cdots + \beta_1S + \beta_0$$

And the observer gain matrix we can find. So here, we are assuming in the control design for pole placement control design, we assume that our matrix, the controller matrix, is in this form, right? We have already done this, and here also we have to assume the observability test matrix. So, if you remember, we had the observability test matrix here. I'm not going to detail the observability test matrix. We had

$$N = [C^T \quad A^T \quad C^T \quad (A^T)^2 \quad C^T \quad \ldots]$$

So here also, we can follow the same process that you are doing. We have done this for the control design. So finally, we have the control observer gain matrix L equal to this structure. You can follow the same process.

$$L = [(\beta - a)N'N^{-1}]^T$$

Okay, so we have

$$\beta = [\beta_{n-1} \quad \beta_{n-2} \quad \cdots \quad \beta_0]^T$$

$$a = [a_{n-1} \quad a_{n-2} \quad \cdots \quad a_0]^T$$

and so this is how we can find the Observability observer gain matrix. I'm not going to develop it, so you can find how it is derived. Let's stop here. In the next lecture, we will take an example and implement the concepts we have covered in this lecture. Thank you.

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