Advanced Aircraft Control Systems With MATLAB / Simulink

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Lecture 11

Observability and Observer Design

In this lecture, we are going to discuss the observability of the system. This is a very, very important part of designing the control, state feedback control. And also, we will look at the mathematical formulation of the observer. Before we proceed to the main contents of the lecture, of the control algorithm we have developed for the plant to be controlled, so let's assume this is the plant we are having, and we have the controller here, and the controller provides the ideal control input to the plant to be controlled.

And we are having x as the plant output, so let's assume this is our $\dot{x} = f(x)$, and we are having the summing point where we get the comparison between the desired values and the actual state x states here. And the sensor provides the states for designing the state feedback control sensor block. Another question is, if you notice, our system output $Y = CX$, right? And this is the error, and also we are designing the controller

$$U = K(X_d - X)$$

Right now, if you notice here carefully, this is the state used to design control u. Now, how are we going to ensure all the states are available for us to design control u? Because if you notice here, we are assuming in our previous examples that all the states are available to design control.

That's why we are saying that this is state feedback control. State feedback control. So, we have to ensure what are the states that are available to us for designing the control. Whether if any of the states are unavailable to us for designing the control, we need to estimate that state. So, if you see this example, so here we are having the system, here we are having the system,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

so this is the dynamic system given to us, and if you draw the signal flow graph for this particular system this is the signal flow graph we are getting, right? So here, if you notice y equal to here x_1 , y equal to x_1, but this y is not in the y equation, we don't have the x_2 parameter, so only if you consider here, for example, so only you are having the x vector and y equal to cx, so y equal to only the x_1 parameter available to the sensor for measuring the state, so that's why x_2 here is unobservable, but x1 is going to be observed by the sensor here, so this is how we need to find the x_2 estimation of x_2 so that we can design state feedback control, so this is the main objective of this content of this lecture, how we can estimate or how we can find all the states available to us for designing control. So now, if you consider our aircraft equation for the rotational motion of the aircraft, for example, the rotational motion of the aircraft,

So, here if you go back to our equation of motion. So, this is the rotational dynamics equation, the first three equations, and this is the rotational kinematics, the last three equations. So, if you notice carefully in the rotational dynamics equation, we are only applying the control which is l, m, n. So, l, m, n can be Together with control plus disturbance.

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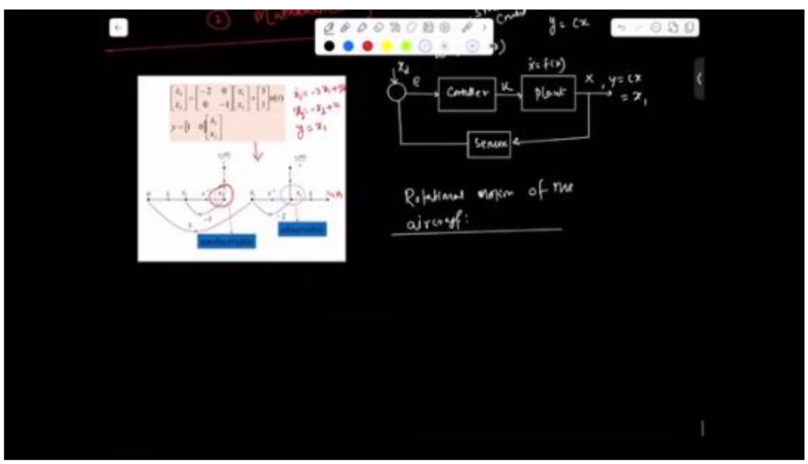

But here basically we are applying the control input. To this parameter. And we are propagating. After applying this control. We are propagating the dynamic equation.

Where actually we are. Finding the state P, Q, R. So this P, Q, R. Going to this equation. To solve.

Phi theta psi. So you can say that. Here. We have the measurement. We have the measurement P, Q, R, maybe using a gyroscope, but we don't have a measurement for the Euler angles phi theta psi.

Phi, theta, and psi. But since P, Q, R, phi to the psi, the function of P, Q, R, I can write phi, theta, and psi, which are the functions of P, Q, R, so we can find phi, theta, psi indirectly. So here, basically, we can say these parameters are affected by the control input, but these are not directly influenced by the control input. So, in this case, we can comment that Phi, Theta, Psi are not directly observable as variables to us. So, estimating Phi, Theta, Psi using the measurements of P, Q, R, we call it, this is actually the process we call observability.

Now, if you define the definition of observability, When an observer estimates the entire state vector, it is called a full-order observer. So, in this case, we can estimate P, Q, R; we can estimate P, Q, R directly from the control equation. We can find P, Q, R if you apply the control input in this equation. We can find P, Q, R, so here we don't need to estimate P, Q, R, right? But we need to estimate phi, theta, psi using this kinematic equation. So this is what we can say when

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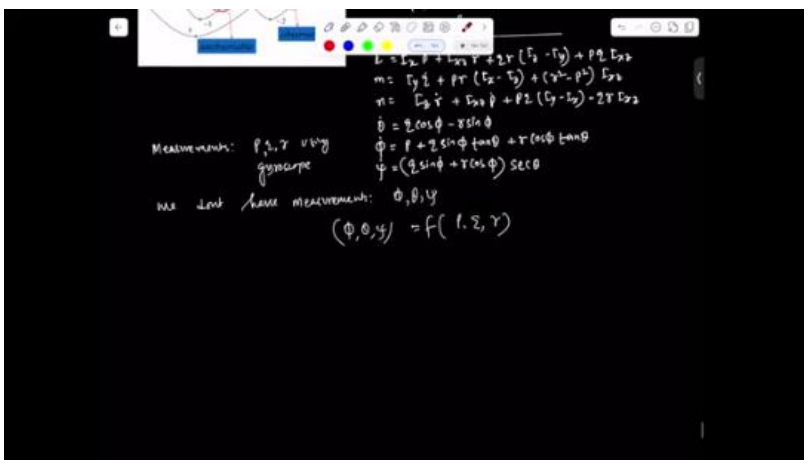

an observer estimates only unmeasurable, so here, basically, we can state we can measure from the given P, Q, R using this equation. So, this is the observable which estimates only unmeasurable state variables. So, for the present case, phi, theta, psi is called reduced And another definition, a controller, a controller which generates control input of a plant based on the basically estimated state vector, the state vector is called the compensator. This is how we can define the definition of the observer and how we can define full-order observer and reduced-order observer and if compensated, so this whole stuff comes under observer design for the controller design. So now we'll go to the mathematical formulation which the observer, how we can test the observability of the system, so generally, the observability of a system we can define based on the unforced system. So our given system is system dynamics, we can write

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

But for the observability test, we only take $\dot{X} = AX$ and $Y = CX$. We don't take any control input in the dynamics.

So, for the observability test, if you want to test the observability of a system, for the observability test, test, we consider $\dot{X} = X$ and $Y = CX$. So, if you notice carefully here, how here we can take a and c matrix for testing the observability because $\dot{X} = AX$, it talks about the state how it is propagated and $Y = CX$ talks about how we can take the states and x is the number of states available in the controller in the plant, right? So now, similar to the continuity test, the observability test, we can do some observability test matrix which we can define as the observability test matrix, test matrix we can define as

$$N = [C^T \quad A^T \quad C^T \quad (A^T)^2 \quad C^T \quad ...]$$

So, if the rank of n is found to be in matrix full rank and we can say the system is fully observable. If you can't, if the rank of the rank of n matrix is found to be less than the order of the matrix, then we can see the system is not fully observable. Now, let us take an example to validate this concept. So, let us assume we have the system matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

We can test the observability of the system. Right. So, let's, this is, if you notice here, the system matrix is 3 cross 3. So, we have to go till, we have to find C transpose, A transpose, C transpose, and A transpose, right, and C transpose. So, let's find this pair of matrices. A transpose, we can say, whether the system is observable. Okay, so here we need to find for c we need to find for c transpose. So, c transpose we can find one two zero one zero zero

A transpose, we can find

$$A^T C^T = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$${A^T}^2 C^T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So we can form the N matrix now. If you substitute all the element terms in the N matrix.

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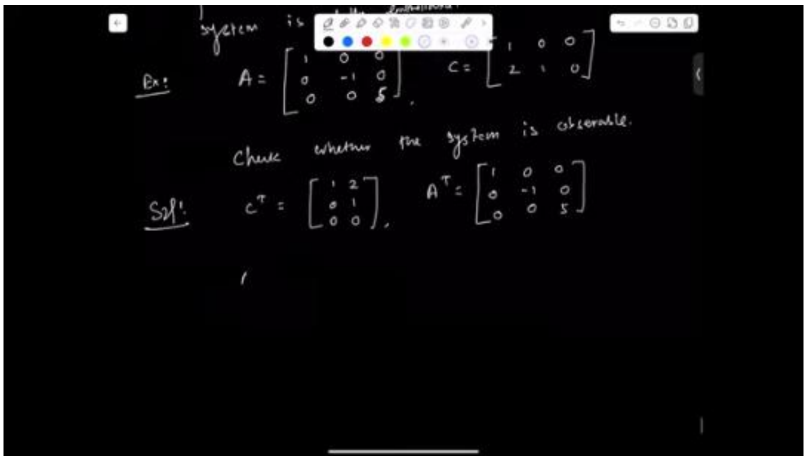

So, this is basically C transpose, this is A transpose C transpose, and this is A transpose square C transpose, right? Now, let us find the rank of this N. The rank of N you can find it comes up to be 2. So, but it means the rank is less than 3, so hence the system is not observable. Observable.

So, one of the states can be observed from the output equation. So, now we will go to the mathematical derivations of how we can find the mathematical formation of the observable test. So, here we are going to consider the system Mathematical formation. We have the system, LTI system,

$$\dot{X} = AX + BU$$

$$Y = CX + DU \dots Eq(1)$$

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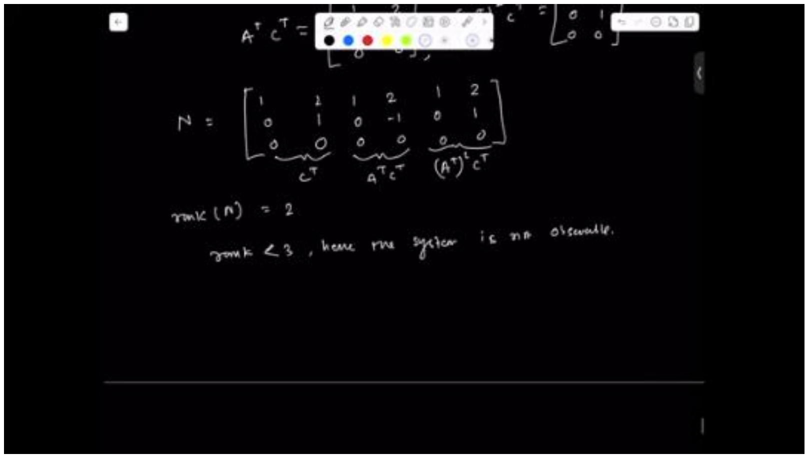

Why is it an LTI system? Because the coefficients of x and u are assumed to be time-varying. So, it is not a function of time, that is why linear time-invariant. Here also, we are assuming the observer dynamics. How this dynamics is found to be, we will explain.

$$\dot{X}_o = A_o X_o + B_0 U + LY \dots Eq(2)$$

So, here o indicates for dynamics and this is the original system dynamics, right? So here L, here L is actually the observer gain matrix, gain matrix. So here the matrices, the matrices A naught, B naught, and L must be selected, selected such that estimation, estimation error S plus 0, we can assume $E_o = X - X_o$

This is the system state, this is the observer state, is brought to 0 in steady state. So, once this condition satisfies, one important note for designing the observer: the eigenvalues of the observer dynamics are chosen to be faster than the front dynamics. So I can write here, for designing why it is so, we'll explain why we'll be taking the example in more detail for designing the the eigenvalues. So here, basically, this is the you can assume that the desired eigenvalues that will be chosen for the observer design must be, must be, must be faster than the plant It means the eigenvalues of the observer should be more negative than the eigenvalues of the closed-loop system.

If eigenvalues of the observer are chosen to be highly negative, then such observer would be highly sensitive to noise. Actually, in practice, observer eigenvalues are chosen so that they are, they are four times, so basically we are four times more negative than the desired closed-loop performance. So here, if we assume the eigenvalues of a should be less than four times Eigenvalues of. So, basically, it is the way I can write desired dynamics or state feedback control and the desired dynamics for observer design.

We'll go through all this stuff in detail while having the example. Now, we'll go to the, let's go to the mathematical formulation of how we can find all the details of the observer analysis. So, we have the system

$$\dot{X} = AX + BU \dots Eq(3)$$

$$Y = CX + DU$$

and we have the observer dynamics

$$\dot{X}_o = A_o X_o + B_0 U + LY \dots Eq(4)$$

So, adding we are doing adding and subtracting $A_o X$ On the R.H.S, okay, let me define another term here. Also, we have

$$E_o = X - X_o$$

$$\dot{E}_o = AX + BU - A_o X_o - B_o U - LY \dots Eq(5)$$

$$\dot{E}_o = A_o E_o + (A - A_o)X + (B - B_o)U - LY \dots Eq(6)$$

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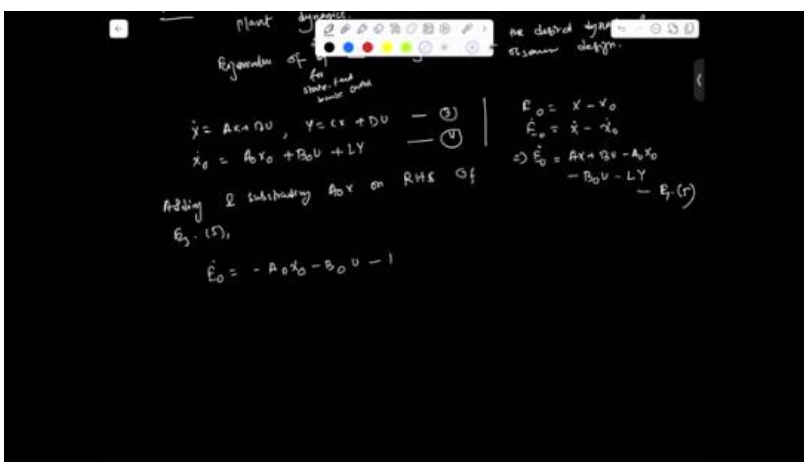

Now, if you substitute it in this equation,

$$Y = CX + DU$$

in equation 6, we have

$$\dot{E}_o = A_o E_o + (A - A_o - LC)X + (B - B_o - LD)U \ldots Eq(7)$$

And if you notice equation 7, E naught goes to 0 in steady state if all the eigenvalues of E naught are negative or stay in the left-hand plane. And at the same time, if these matrices are zero. Okay, so I am repeating again to make E_o go to zero. So, all the eigenvalues of A naught should be negative, and this matrix and this matrix should be zero at the same time.

So, we can write E tends to zero as t tends to infinity if the eigenvalues of A naught are negative and

$$A - A_o - LC = 0$$

$$B - B_o - LD = 0$$

So, if these conditions are satisfied, and at the same time, you can write from this equation $A_o = A - LC$ and $B_o = B - LD$ okay? This we can write from this equation. And from this equation, we can write right now. And if you substitute these conditions. In equation 7, what do you get? So, substituting the conditions in equation 7, we have

$$\dot{E}_o = (A - LC)E_o \ldots Eq(8)$$

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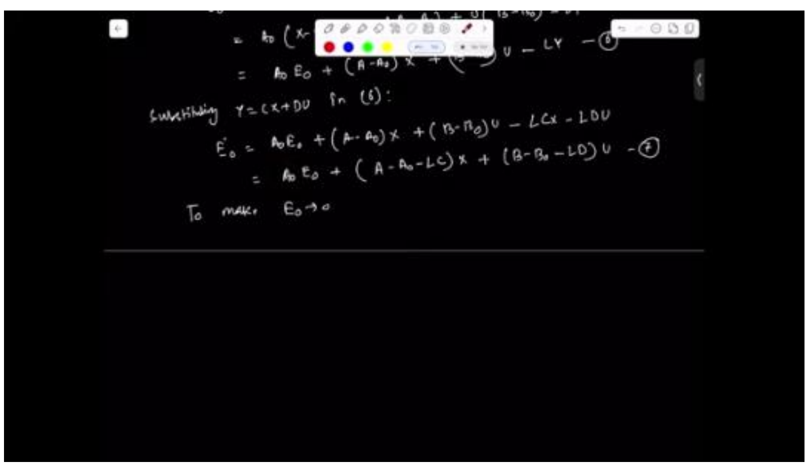

Now, if you substitute, if you again, if you substitute this term, if you substitute this term, this condition in the observer dynamics, what is written here. So, substituting

$$A_o = A - LC$$

$$B_o = B - LD$$

in equation in equation 4, we will get full order observer in the equation

$$\dot{X}_o = AX_o + BU + L(Y - CX_0 - DU) \ldots Eq(9)$$

and also you can simplify this expression

$$Y - CX_0 - DU = CE_0$$

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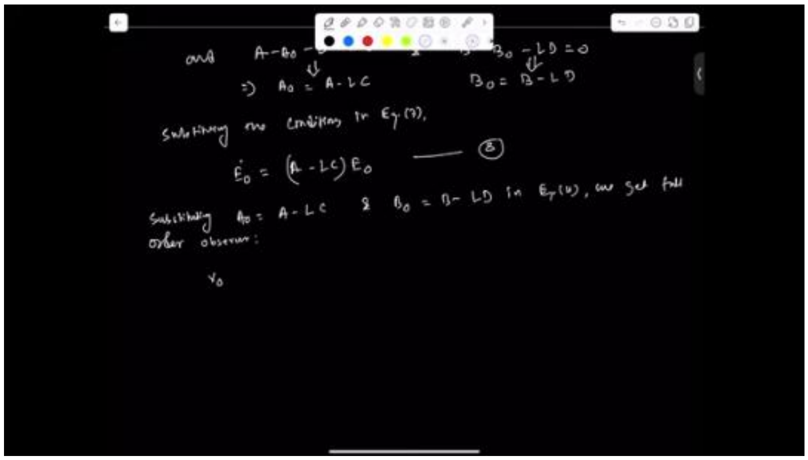

So here, the term here we call this term, we call a residual term, and we can represent as so. This is basically a residual. So, if you notice carefully, if a residual term goes to zero, if error dynamics $\dot{E}_o$ goes to 0.

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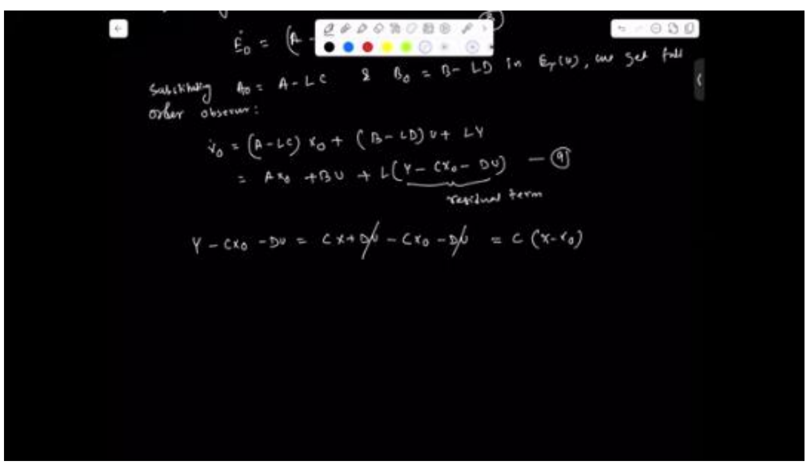

So, this is the condition, right? And as E tends to infinity. So, in this condition, and hence we can say the system is asymptotically stable. So, this here, the system is nothing but the estimation error dynamics. So, estimation error dynamics.

So, this is how we can design the observer dynamics or estimation error dynamics and how we can validate. Uh, how many states are available in the output, which will be used to design the state feedback control, and this is the full mathematics to come out with the error between the actual and the observer values from the sensors. Um, now if you want to see the closed loop diagram So, we can draw the closed loop diagram of the system. So, here if you consider Y=CX, sorry. So, let us assume the most practical one. So, here we can write the system as

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

x dot equals to ax plus bu. We have the d u part and the observer dynamics. Let us assume the observer dynamics here. This is the observer dynamics. From this, we can write from equation 9:

$$\dot{X}_o = AX_o + BU + L(Y - CX_0)$$

If you assume DU part is not there, this DU part. So, we are making some simplification in this dynamics. So, here l into we can write $Y - CX_0$, right? Now, if you want to draw this control diagram, how the system is, it can be written in the closed-loop block diagram.

So, we have the B matrix. First, let me draw the figure B, coming point. We have the integrator 1 upon S, this is the C matrix. And we are having the feedback here. A. This is X. This is $\dot{X}$. Let's assume we are having U here.

And this is the feedback. And this is positive, right?

$$\dot{X} = AX + BU$$

$$Y = CX$$

Now, here also you have the observer dynamics, yeah. So, here we are taking the output here and we are having a comparator. One comparator compares the output from the original system and we have the observer gate matrix here or L. This is coming here and this goes to another summing point b and this is the control is coming here. So, this part I am, you know, drawing based on this dynamics, ok? And we are also having

this. We are having another block 1 upon s. This is $\dot{X}_o$. This is X_o and we are getting multiplied by C, which is giving me Y observed y output and this is negative. We are also having an A matrix here. A is here is coming from the X_o and we are having feedback here, plus. So, here if you tell you with the system, so $Y_o = CX_o$. So, let me define it. So, here first I will find this term L into $Y - CX_o$ and this is the B U term which is going to extra dynamics. And also we have AX_o. So, this is basically we can say a full-order observer figure, a full-order observer.

So, this is how we can come up with the full observer for the system, and we can see how we can do it practically. One example is how we can do the observability of the system, and how we can come up with the aerodynamics, and how we can find it because here the most critical part is finding this L, the observer gain matrix. So, we will do all this talk in our next lecture. Thank you.

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