Introduction to Finite Volume Methods -I
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Lecture – 37
Properties of matrices-II

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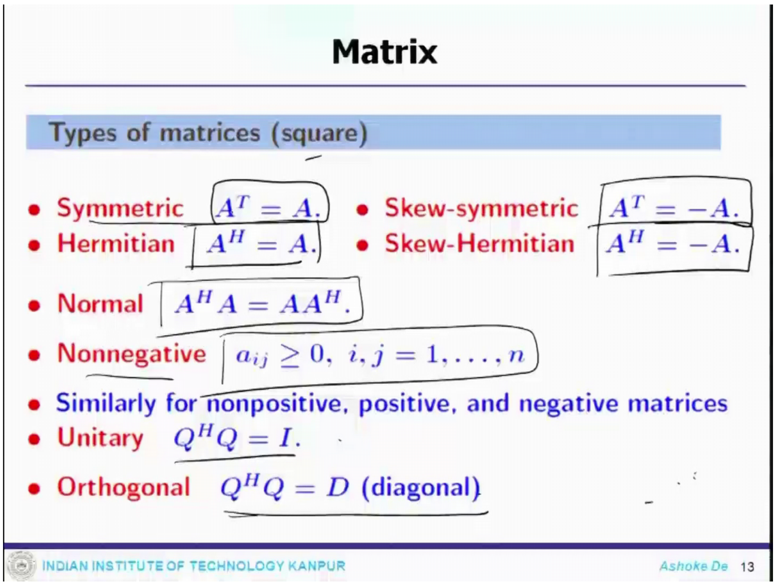

So, welcome to the lectures of this Finite Volume Method and what we are talking about here this is we are in the process of getting a solution for the linear system. And before doing that or talking about the linear solver what we have chosen to do is that discussion on some properties of the matrix and that is what we are doing. And till the last class what we have done, if you look at it this is where we are talking about the properties of this different matrices like symmetric matrices, skew symmetric matrices, normal matrices, nonnegative all these that is what we stopped in the last class.

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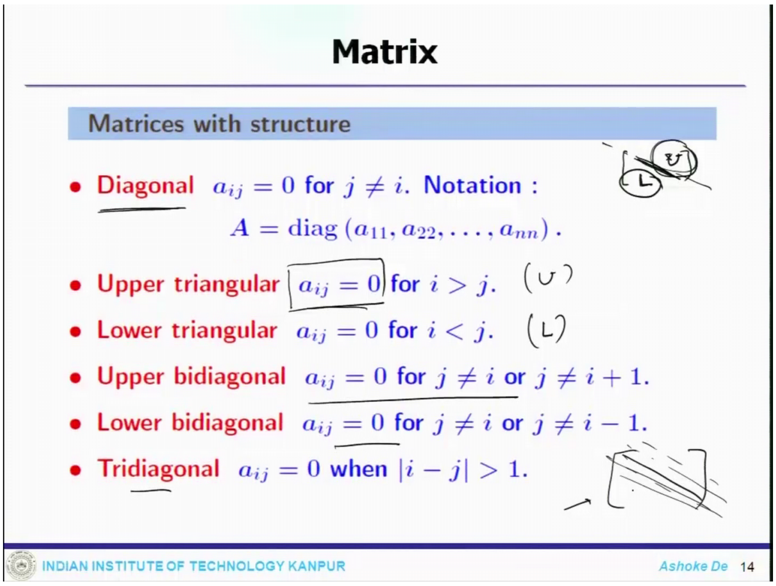

Now, moving ahead when we call the matrices see this portion of the properties and all these you can find it any text book. So, that is not an problem. So, just, but few important properties which are going to be crucial that is what we are discussing here and then we will move to the calculation of the linear solver, like one of that kind of property is your important property is the diagonals. So, diagonal means when you talk about diagonal of A you only refer to these elements. So, these are the diagonal elements and anything upper that it is a upper diagonal anything lower that it is a lower diagonal. And also this is called, if you only consider the upper half of the matrix this is called the upper triangular matrix.

And these are all notation or the mathematical notation how you define. So, you may not need to worry about that much, but what you need to understand here is the concept of that. Because these are the things like diagonal element, lambda, eigenvalues, spectral radius, singularity, nonsingularity, conjugate, transpose, determinant, lower triangular these are going to be used extensively while talking about the linear solver. So, it is very much important that you get an hands on idea about these system. So, when you consider the upper half of its, it is called the upper triangular when you only consider the lower half of it, it is the lower triangular. Then you can have upper bidiagonal which you can be represented like that you can have lower bidiagonal and you can have tridiagonal

Tridiagonal means, you have one diagonal then one up diagonal like that. So, this is called tridiagonal system if you have two more then it is called pentadiagonal system. So, these are the system other way also you can think that these will be looking like an banded system and when we just had an brief idea discussion about direct method or iterative method we also said that the matrices are going to be primarily sparse and non-singular. But they can be sometime tridiagonal or pentadiagonal system; that means the banded system.

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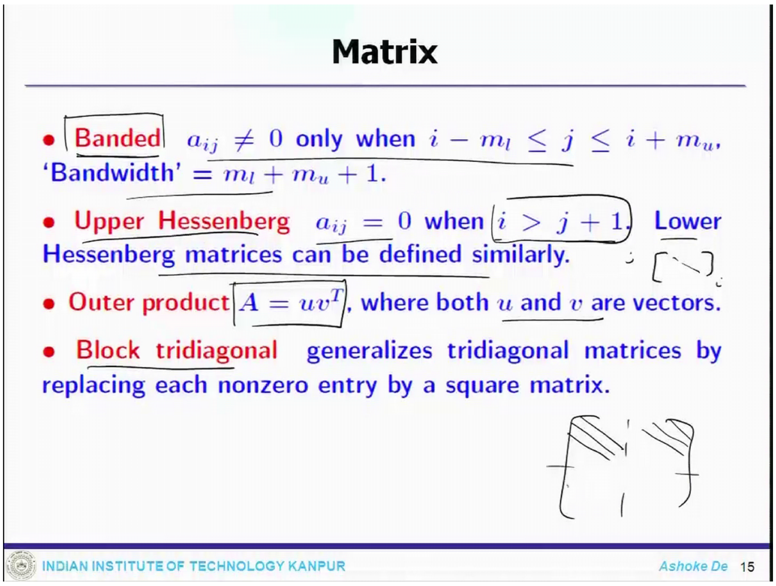

So, just moving ahead to that this is what the definition of the banded system. So, this is a mathematical definition which you can find it any linear algebra book, but what is important again is the idea of the banded system. Now, then another system is that it is called the upper Hessenberg matrix, that is when the components are 0 for i greater than j plus 1 and the lower Hessenberg matrices can be defined similarly. So, which will be the reverse of that and; that means, in a particular system you have upper Hessenberg, you have lower Hessenberg and that can be constituted using your i and j elements then outer product.

Outer product means if you have two vectors u and v then the matrix will get you the outer products or the then you can have also block tridiagonal system that generalize the tridiagonal matrices in a system. So, block tridiagonal means I can have block tridiagonal systems. So, it could be one tridiagonal system here, you could have one tridiagonal system here you could have one tridiagonal system here, these are called the block tridiagonal system.

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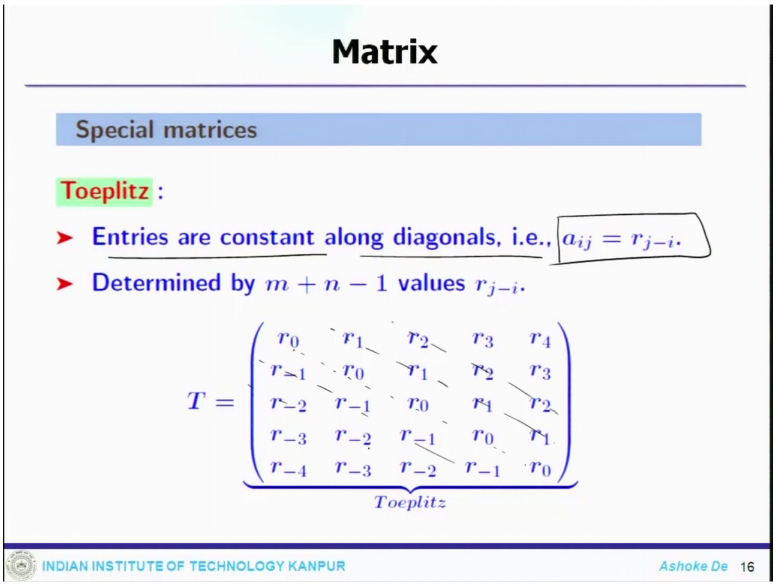

Now, special matrices Toeplitz it is the entries are constant along diagonal; that means, a ij equals to r j minus 1. So, along the diagonals if you look the entries are constants. So, if you look along the upper diagonal, if you look along the second upper diagonal they are constant. So, this is called the Toeplitz and it can be determined by the mathematics like that.

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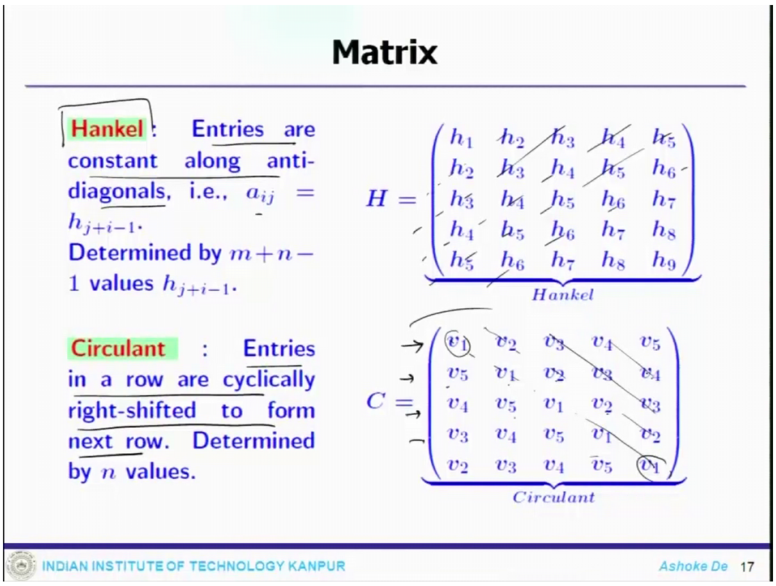

Now, also Hankel that is a another matrix when the entries are constant along anti diagonals; that means, a ij equals to h i plus j or j plus i minus 1; that means, all the entries which are going to be constant along anti diagonal terms. So, anti diagonal terms means the, these are the component sitting in the diagonal. So, these are going to be the anti diagonal terms and you can see that they are going to be constant then circulant, here entries in a row are cyclically right shifted to the to form next row.

So, these are my diagonal elements and entries in a row are cyclically right shifted to form a next row. So, this is the first row which are having v 1, v 2, v 3, v 4, v 5. So, cyclically shifted means my v 5 will come here. So, that is how it goes v 5 v 1 v 2 v 3 v 4 second v 4 moves here v 5 v 1 v 2 v 3 then v 3 moves here. So, finally, you get v 1 and again at the end v 1. So, in that process you end up getting these are similar values. So, that is called circulant.

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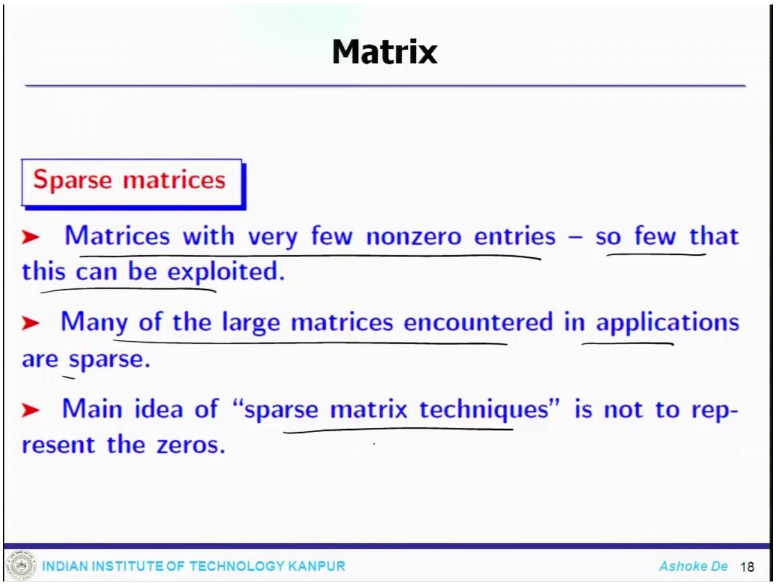

Sparse matrices: so, the matrices with very few non-zero entries. So, few that this can be exploited many of the large matrices encountered in real applications are sparse and main idea of sparse matrix techniques is not the represent the 0's it is basically having the few 0's, but they are randomly located.

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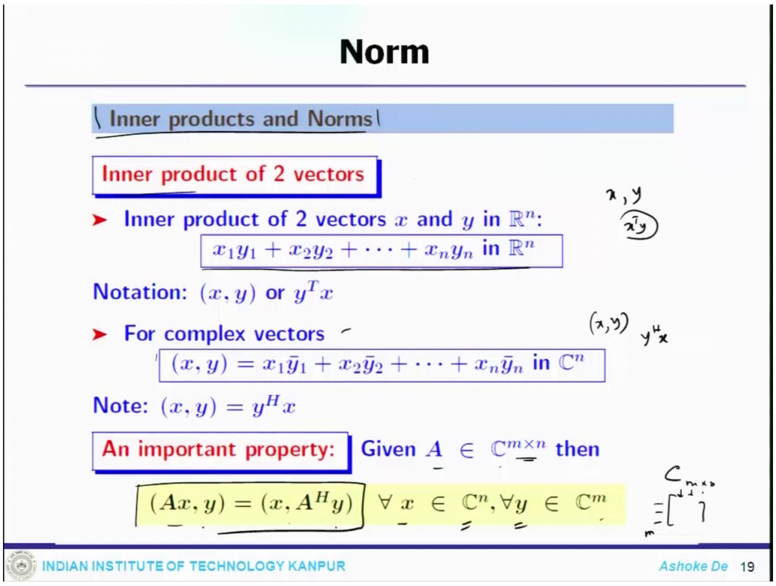

Now, coming back to calculation of norm; how one can actually calculate the norm of a system and how do you do that it is having an inner product and norms the inner product of 2 vectors. So, if you have 2 vectors like x and y then the inner product which essentially going to be the x transpose y and if you write in that. So, it is how you are going to get the inner product of that, that essentially the dot product of two different matrices.

For complex vectors this is going to be the inner product. So, that either one can write y H x that is the inner product of x and y. So, property which is very important is that for a a which belongs to a complex m bar plus n system; that means, m rows n columns A x y would be x H Hermitian y. So, that is how the product can be defined for all x which belongs to C n for all y which belongs to C m. So, c C a m by n system. So, these are m and these are n.

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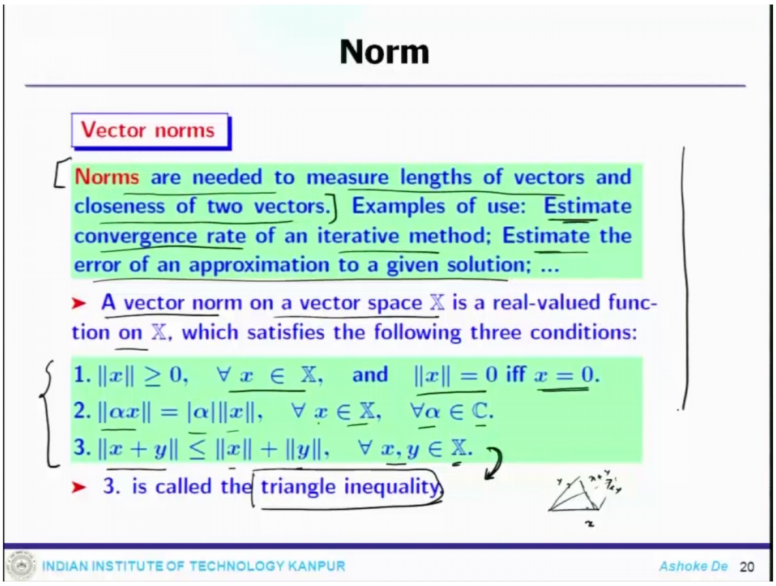

Now, once you get the inner product of two vectors now you can actually estimate the vector norms and how you get that the norms which are needed to measure lengths of vector and closeness of two vectors. For example you can estimate the convergence rate of an iterative method, estimate the error of an approximation to a given solution. So, the important definition of norm is that it is an measure of length of vectors. So, when we talk about the iterative method we try to find out the convergence rate. So, that is where actually you can estimate it when you talk about some error you can also estimate that. Now the second important point is that a vector norm on a vector space X is a real valued function on X which satisfied these three conditions.

Number one, the magnitude must be greater than equals to 0 for all x belongs to that vector space and the magnitude of that would be 0 if and only if the vector is 0. Then if you multiply it with any scalar alpha and take the mod this can be represented as mod alpha into magnitude of x. For all x belongs to this vector space and for all alpha belongs to any number then the x plus y the mod of x plus y which is an inequality must be less than mod x and mod y for all x and y which belongs to the vector space.

And this property which is essentially the inequality property is called the triangle inequality, why? Because if you have two vectors x y you can always form a triangle and try to find out so, the property. So, this is y it could be x plus y. So, that is what called the triangle inequality, these are some important information or some important definition that one has to know before we do some detailed discussion on any linear system for linear algebra.

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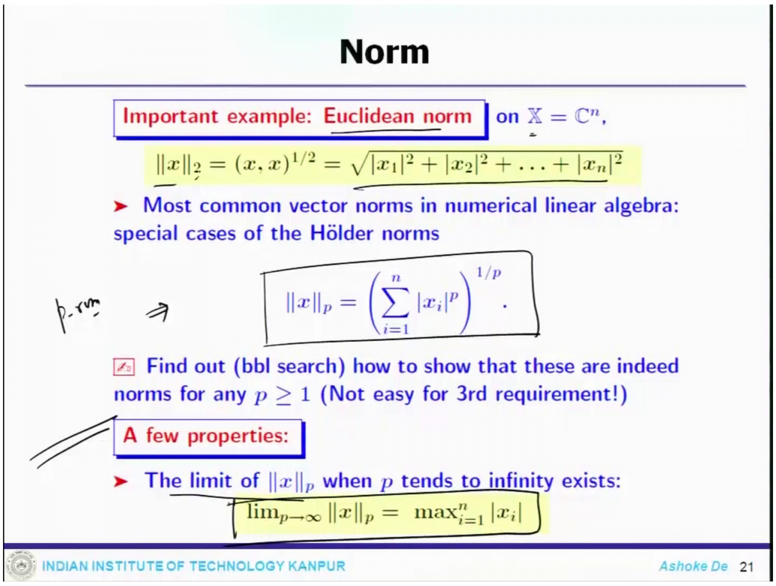

Now that brings to that Euclidean norm; so, Euclidean norm on a vector space of n C n. So, this is called the l 2 norm or Euclidean norm. So, this is square root of all the length square, this is how one can find out. Now, the most common vector norms in numerical linear algebra special cases holders form like this ok.

So, you can show that the important property is the limit of this mod x p when p tends to infinity it actually exists. So, this is one of the important property for this p norm, this is called the p norm and when p equals to 2 we can find out n 2 norm. So, different norms can be obtained like that now.

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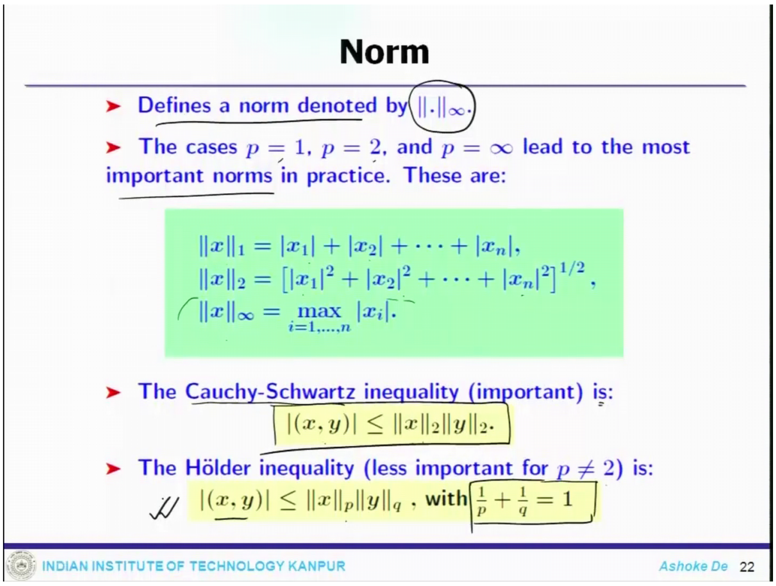

So, you define a norm denoted by the product with infinity, as I said if p equals to 1, p could be 2, p could be infinity lead to the most important norms in practice. So, p 1 means you get x 1 x 2 x 3 like that. When p equals to 2 you get only the l 2 norm which is the square root of x 1 square x 2 square x 3 square and this infinity you got the maximum of that. Now, there is one when you talk about the norm immediately this brings to an important information about Cauchy Schwartz inequality.

So, this one has to also know and the property of that is that if you have two vectors norm x and y I mean at p 2; that means, l 2 norm the I mean the dot product or the inner product of x and y must be less than equals to x 2 and y 2 separately. So, this is another property of the inner product of these norms. Now, similarly the holder inequality which is not that important for p not equals to 2 that is that x y mod of x y inner product less than equals to x p y p with the condition 1 by p plus 1 by q equals to 1. So, that is another property that can be used for the information that in the system of the linear system

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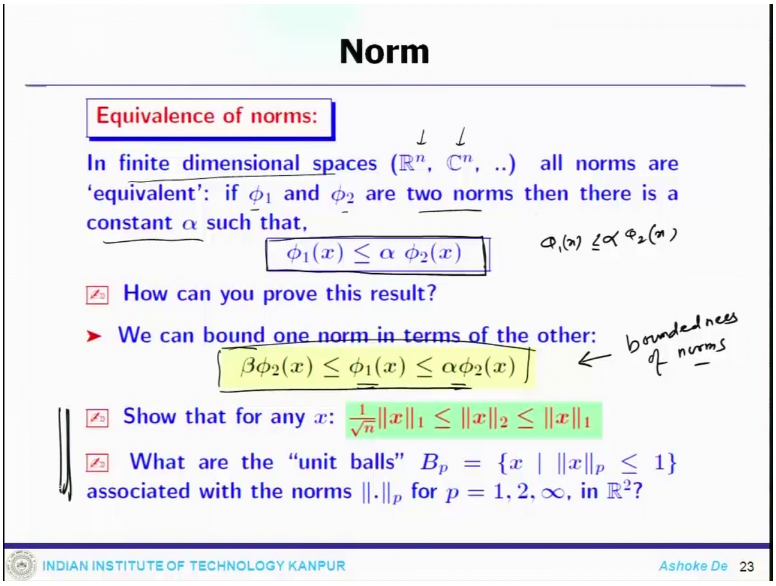

Now, one can find out the equivalence of norms, how one would do that? In a finite dimensional space let us say you have R n C n all these are finite dimensional space all norms are equivalent. For example, if phi 1 and phi 2 are two norms then there is a constant alpha such that phi 1 x less than equals to alpha into phi 2 x. So, that gets you an important property of equivalency between these norm. So, I have two different norms phi 1 and phi 2 and the equivalency can be obtained by there is a constant alpha such that phi 1 x less than equals to alpha phi 2 x.

Now, one can you can prove these things. So, that is one has to look at in the linear algebra, now we can bound one norm in terms of the other also. Similarly if you have a scalar multiplied with phi 2. So, beta phi 2 must be less than equals to phi 1 x which is also less than equal to alpha phi 2 x. So, this is the sort of bounded ness of norms. So, the norms can be also made bounded so, you can get these properties. So, these are for some homework information one can do that.

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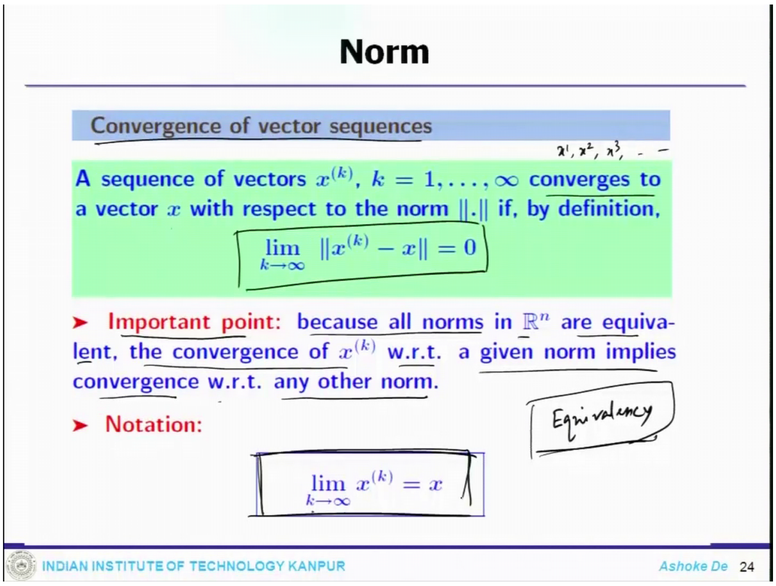

Now the important information which becomes is the convergence of vector sequences. So, this will be an one of the information that would be again used for when discussing about the linear solvers. So, a sequence of vectors for example, x k k 1 to infinity; that means, you have x 1, x 2, x 3 like that. So, you have a sequence of vectors which will convergence to a vector x with respect to the norm product or norm dot, if by definition when x k minus x or the magnitude of that when k tends to infinity is equals to 0; that means, in a limit this guy becomes 0. So, that is where you can find out the convergence of a sequence of vector.

Now, the point here is because the or rather the important point here is that because all norms in R n are equivalent the convergence of x k with respect to a given norm implies a convergence with respect to any other norm. So, which will be written in terms of like this, the limit k tends to infinity x to the power k equals to x. So, there is a so, as we go along with this properties and the norm calculation one after another properties use the previous information. So, here the property which was used is the equivalency. So, you can see that the convergence property uses the equivalency property of the norm and that is why you can write this notation and say that the convergence can be obtained.

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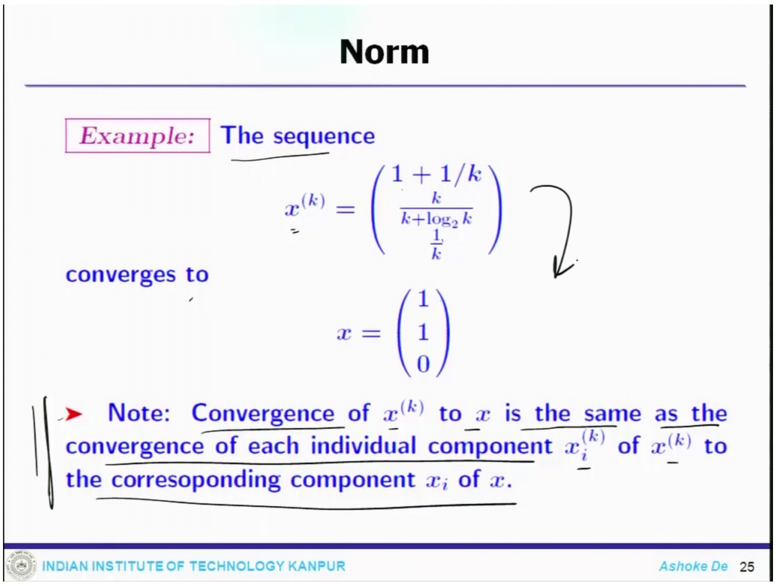

So, that is a example you can see the sequence which is the vectors are 1 plus 1 by k second vector is k by k plus log k third one is 1 by k this will converge to 1 1 0 and the convergence of x k to x is the same as the convergence of each individual component x i k of x k to the corresponding component x i of x. So, this you can see how the equivalency property is used to look at the convergence of this system.

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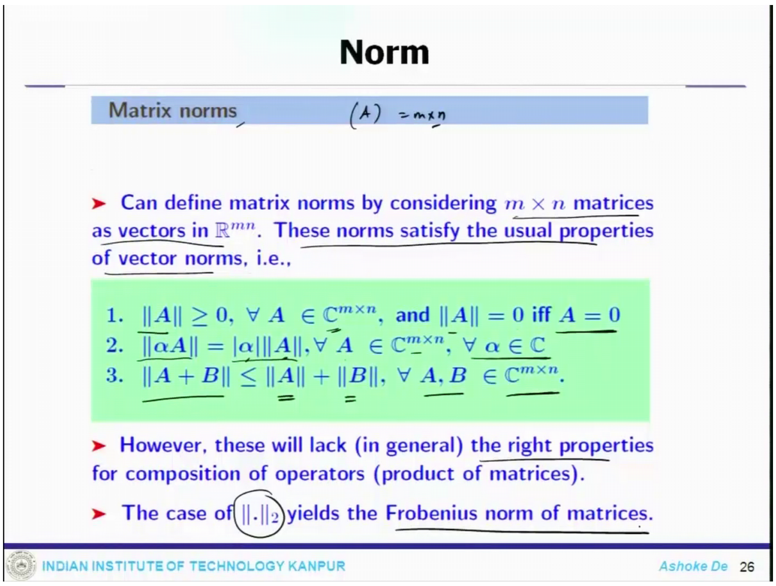

Now, you come down to so, once we talk about the vectors. So, we talk about the vectors norm and we come back to the matrix, why? Because matrix are nothing, but the this is m by n matrix. So, matrix are nothing but the combinations or the metric spaces spanned by the column vectors and the row vectors. So, it has the column vectors and the row vector. So, we can find out so that is what we first discuss the norm of a vector and now we want to look at the norm of a matrix. So, one can define the matrix norm by considering an m by n matrices as vectors in R m n then this norm satisfy the usual properties of vector norms. Which means; those three properties that we have discussed for the vector norm they are going to be exactly satisfied for this case also. For example, the magnitude of A must be greater than 0 for all A belongs to the space in m n and the magnitude of a would be 0 if and only if all A is 0.

Secondly if you multiplied a with A scalar and take the magnitude then it can be equivalently represented as mod alpha and magnitude of a for all A belongs to C m n for all alpha belongs to C. And third which is an inequality again the magnitude of the additional or addition of the addition of these two vectors must be less than equals to mod A and mod B for all A, B belongs to C m n. However, in general, this will lack the right properties for composition of operators. So, essentially the product of matrices so the case where the dot product that yields the Frobenius norm of the matrices.

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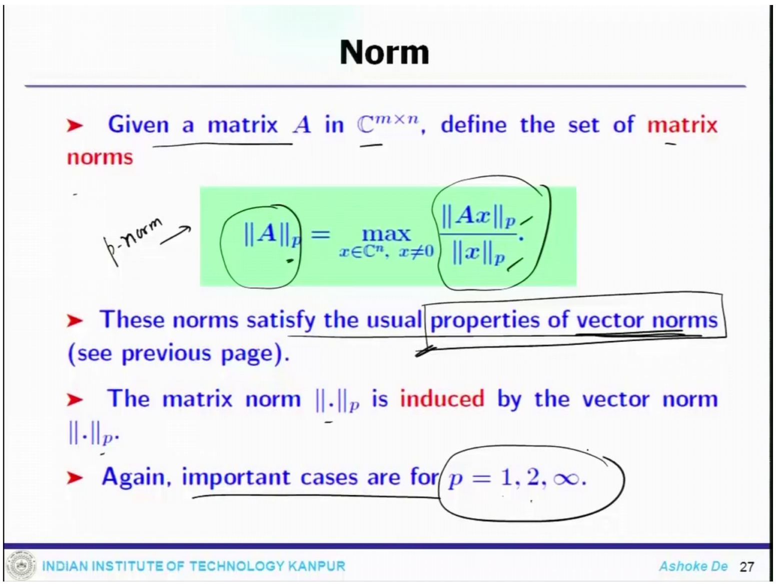

So, now once we define this matrix norm we can see how the other informations can be obtained for example, we have given a matrix A in C m n defines the set of matrix norm. So, for mod A it is a p norm for matrix. So, we are trying to find out the p norm which will be the maximum x belongs to C n where x not equals to 0 is A x the modulus of p and mo x p. So, that ratio is going to give me the p norm of this particular matrix as we have obtained for p norm for vector, now we are using or expanding that same concept for getting the matrices. Now, these norms what you obtain for the matrix they must or they satisfy the usual properties of vector norms so; that means, what we have discussed for the vectors norm they do satisfy the properties of that.

That is one of the added advantage and why that happens again matrix is nothing, but the combination of your row vectors or column vectors. So, that is why it always satisfy the properties of the vector properties. Now when the matrix norm p is induced by the vector norm this, again important cases are for p 1, p 2 p infinity. So, depending on the values of p different norms like for vectors we have obtained in this case also you can obtain those norms for p 1, p 2 and infinity.

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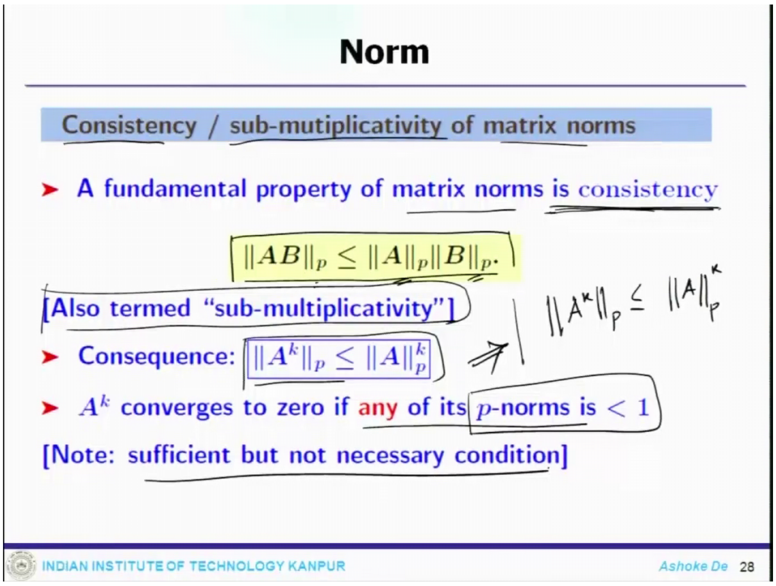

Now, the thing which comes to the table is that consistency. So, that is another important information that one has to know or the sub multiplicativity of matrix norm, what do they say? The fundamental property of matrix norm is consistency; that means, it has to be consistent and that pretty much applies to even our numerical system. You want to device a system or numerical algorithm or the technique which you want to be consistent. You do not want anything which is not non-consistent and once that has to happen some parts bits and pieces also need to be like that.

So, the properties of the matrix norm is also going to be consistent and that can be mathematically expressed like that modulus of AB for p norm which is an inequality less than mod p and mod B p. So, this is called sub multiplicativity. So, this is also known as sub multiplicability. So, the consequence of that it leads to that A k p less than equals to modulus of A k p so; that means, I have A k that p norm must be less than mod A k p. So, that what you get.

Now, A k will converge to 0 if any of it p norm is less than 1. So, that is an very important criteria or constraint, but this is an sufficient, but not necessary so, one note that it is sufficient, but not necessary condition. So, there are conditions which has to be also necessary.

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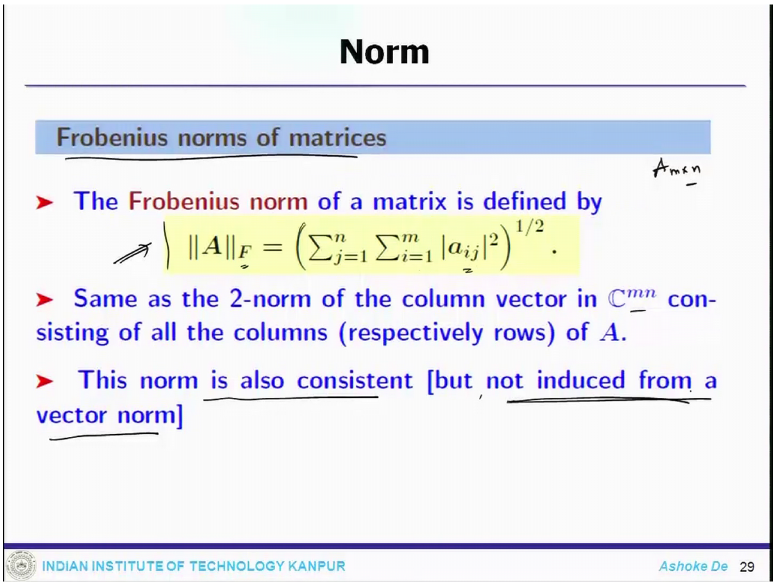

Now, when you look at the Frobenius norms of matrices, the Frobenius norm of matrix can be defined like modulus A F which is F stands for the Frobenius norm equals to j equals to 1 to n i equals to 1 to m alpha i j these are the component of the matrices and the square and the 2 complete term is taken under square root of that.

This is same as the l 2 norm of the column vector in C m n or consisting all the columns of A; that means, A has all this A is a m by n system. So, it has all this n columns. So, the l 2 norm of column vectors are going to the represent the Frobenius norm and the property is that it is also consistent, but it is not induced from a vector for norm. So, the other case it was other way around. So, in this case you can note that this is not induced from a vector norm.

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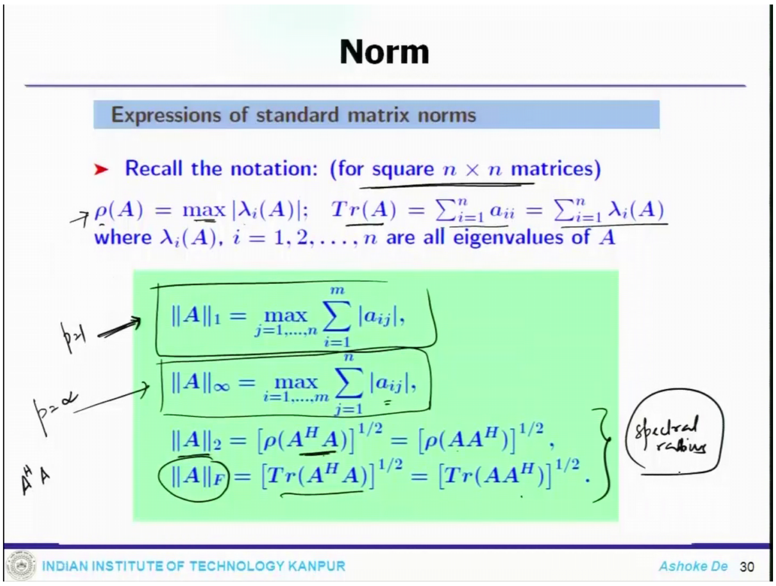

Now, moving ahead you can look at these different matrix norms; that means, when this case it is p equals to 1 this case p equals to infinity. So, the notation is that you have a square matrix n by n, if you have a square matrix n by n rho a is the spectral radius which is the maximum of the eigenvalues of A. So, you find out the eigenvalues of a that is what one can represent lambda i A; that means, it gives you back all the eigenvalues that is there including the repetitive one and the maximum one that is going to be the spectral radius. Trace of A which is the summation of diagonal components or summation of all the non repetitive lambdas.

Now, you look at the p equals to 1 norm. So, this is going to be the maximum of all these elements goes over the loop, when you look at the infinity norm mod a infinity is going to be the maximum of I to m and j goes to 1 to n is the a i j the magnitude. So, one can represent that the l 2 norm is the spectral radius of a Hermitian a and the square root of that. So, whatever matrix you have the Hermitian matrix and this matrix the some sort of an dot product and square root of so, which can be represented as spectral radius of A A Hermitian and the square root of that.

Similarly, if you find out the Frobenius norm which is going to be the trace of A Hermitian A and the square root of that or trace of AA Hermitian square root of that. So, these two are sort of important informations where l 2 norm and the Frobenius norm you can actually represent with the spectral radius. So, using that information you can represent this in this particular format. So, we will stop here and.

Thank you.