**Introduction to Finite Volume Methods -I**
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**Lecture – 36**
**Properties of matrices-I**

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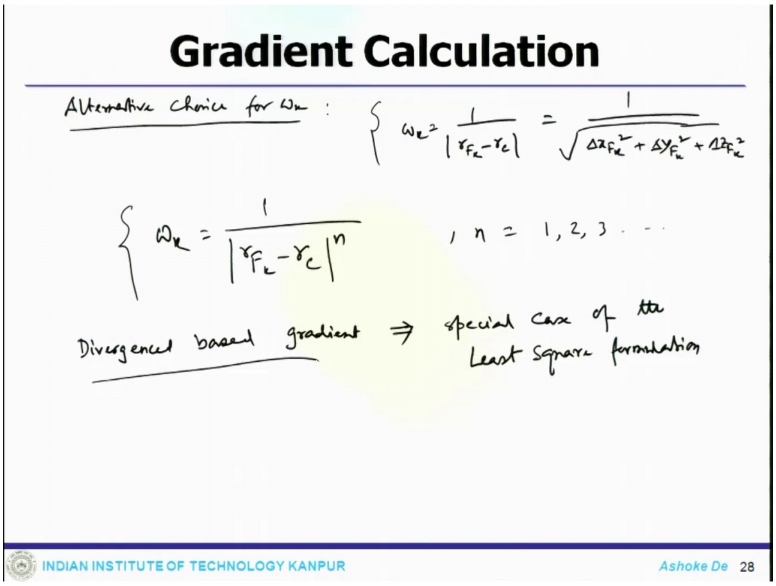


So, welcome to this particular lecture on, we will continue our discussion what we have been doing so far.

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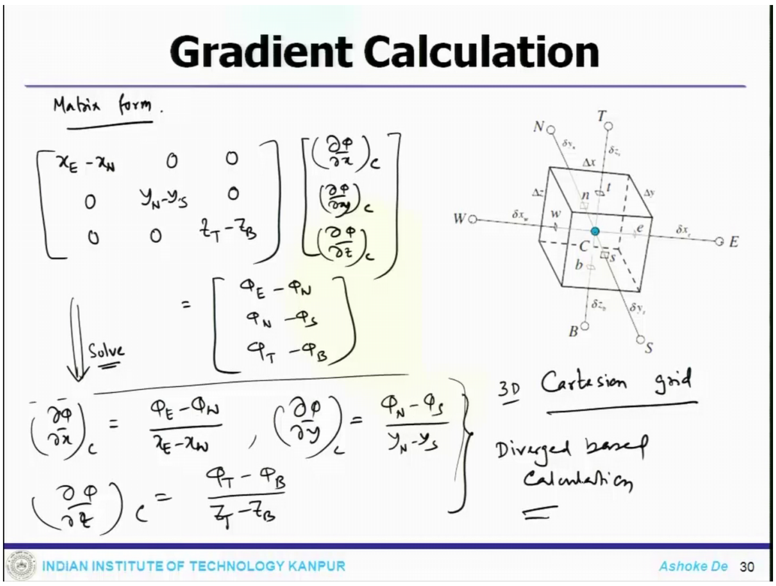

Let us take an simple Cartesian stem cell. So, it is a Cartesian grid, let us consider that and C is the centre of that cell as usual in our convention there is a east cell, this is the west cell, there is a north cell, south cell top and bottom. Now, this is a three-dimensional Cartesian cell. So, that is why you have 6 different phases. So, the number of cells which are surrounding also we will go up. Now, in this case what we can write from our previous matrix equation, so the matrix form we will reduce to something like this.

X E minus x W other component would be 0 this will be 0 y N minus y S 0 this will be 0 z T minus z B that is the first part of it, then we have del phi by del x C del phi by del x del y C del phi by del z C. So, that is there and right hand side would be phi E minus phi W phi N minus phi S phi T minus phi B, if you solve this above equation. So, solving this what you get del phi by del x c is phi E minus phi W divided by x E minus x W and del phi by del y at C is phi N minus phi S divided by y N minus y S and similarly del phi by del del z at C phi T minus phi B by z T minus z B.

So, these are the exactly the values which you have obtained from divergence based calculation. So, when we have done the divergence based calculation we obtain the exactly the same value. So, one can show that this is a special case of least square approximation; also it can be demonstrated that the accuracy of this resulting gradient is at least first order.

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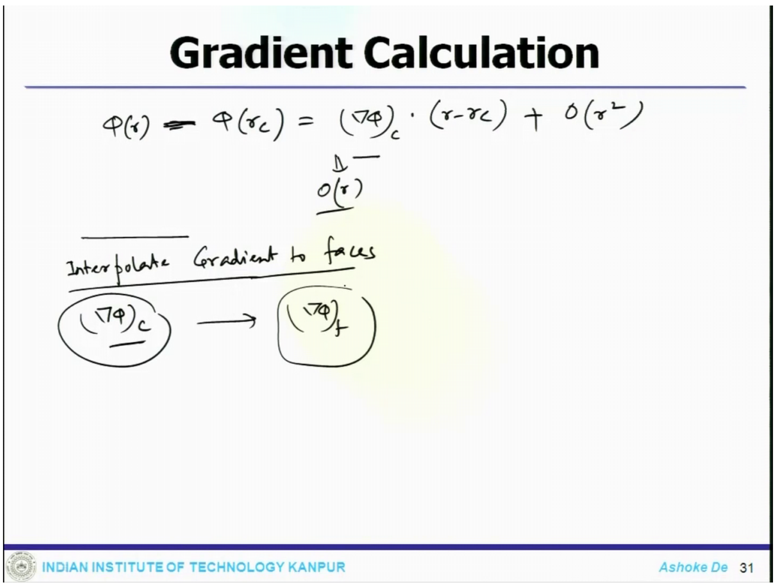

So, one can write down the Taylor series expansion for phi r minus phi r C equals to delta phi C dot r minus r C which is order of r square. So, when solved for these this would be some sort of a order of r. So, at least it is in accuracy in that sense.

So now, the other point comes here is the that how do we interpolate gradients to face. So, far what we have doing is that we are doing the calculation for the at cell centre. So, we are getting that gradient calculation at the cell centre; now important point is that how we transform them back to the cell faces. So, what we have looked at in the diffusion equation is that non orthogonal grids require some sort of a correction term involving the gradients at control volume faces.

So, situation the gradients need to be interpolated from the control value centroids where they are computed to the contour volume faces they will be used. So, from here it can be transferred to the face. So, that is another challenge how do we do that. So, one can look at that some simple stencil.

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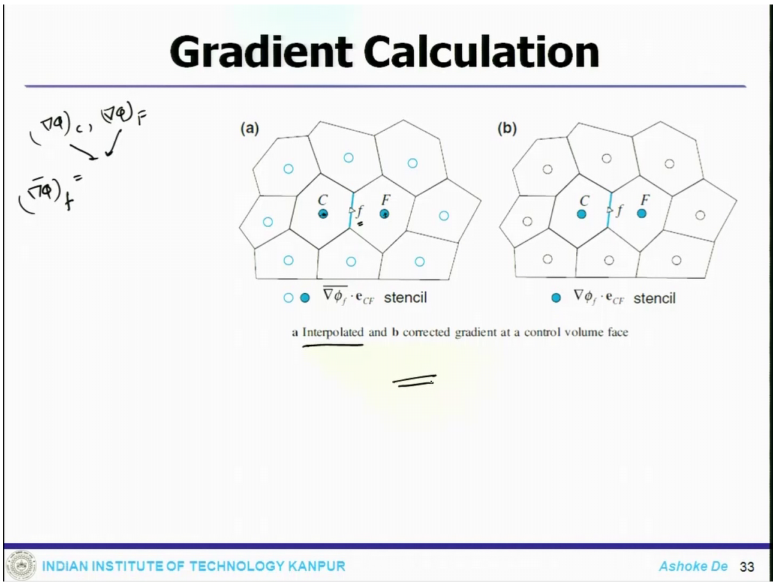


So, this is what one stencil where it is interpolated stencil; that means, I have the cell centre and then I want to calculate at the face and the other case is that corrected gradient at the face. So, once you look at these two, this gradient of the face where the same stencil. I mean the C and F in the surrounding elements if you look at it they are having the same stencil, but ideally they should be same, but when you do the calculation some sort of a corrections is required if there is a some non orthogonality exists.

So, one can see how these delta phi C delta phi F they are actually uses the information at the cell nodes one can be calculated. So, the interpolated flux would be at the face and that can be obtained some sort of an averaging of cell centre C. So, essentially using these two information it can be obtained, now it is vary also much important to look at.

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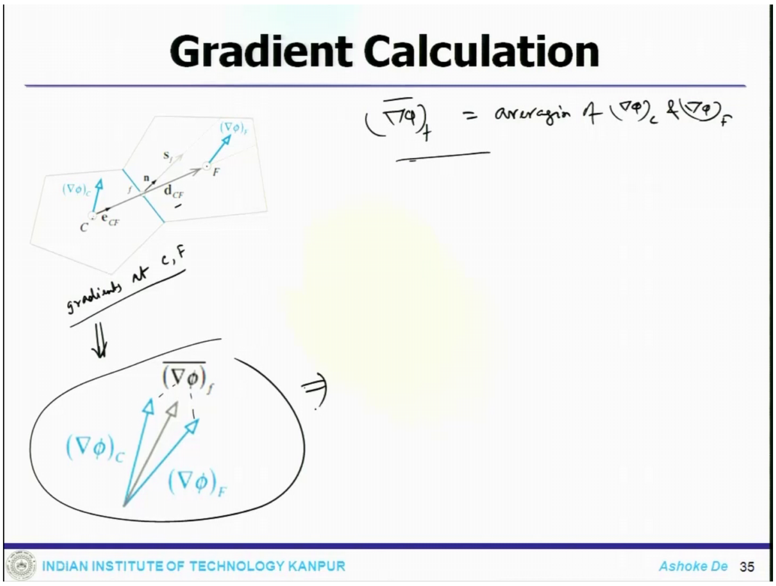


So, this is the stencil and then if you look at this particular figure where you have this essentially, you have this particular first figure what you have the two nodes so, the gradients at C and F.

So, you have these two where the cell centre gradients are available and then when you want to calculated the flux. So, delta phi at face some sort of a averaging of delta phi C and delta phi F. So, that can be obtained using by some sort of an averaging procedure. Now, how you obtain that you look at that the gradient vectors they are in this picture if you take this from here to here and if you just draw the vectors here. So, the vectors actually pointing the, these two vectors and then you can join them and get the averaging of these three, these two and then you can obtain that.

Also it is little bit important for this stencil of the gradient at the face to be heavily based on the nodes which are connected between them. So, there are normal surface vectors, distance vectors and all these are become important, and how?

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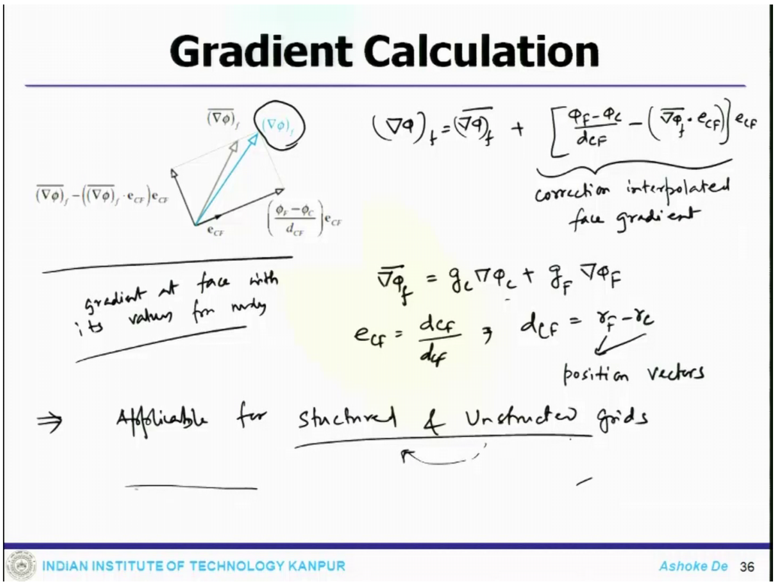


If you put these things in that vector form you can actually get this kind of a gradient at face with its values from nodes. So, this is how you can look at that triangle and by having that vector algebra you can find out the delta f at face. So, if I have to expand this one mathematically how do I write that? This is essentially delta f at face plus there would be some sort of an phi F minus phi C divided by d CF minus delta f dot e CF and then e C F.

So, this is some sort of a correction so these correction interpolated face gradient. So, where you can have delta phi f equals to g C delta phi C plus g F delta phi F and e C F would be d C F by d C F and d C F is r F minus r C. So, one can find out this calculations using these expression and then using the information of delta phi C and delta phi F some error calculation and these two are essentially the position vectors. So, this how you can actually obtain. So, important thing is that this process, this is applicable for both structured and unstructured grids.

Having said that again you can recall the unstructured grid or the structured grade is a special case of unstructured grid. So, one can always come back and the formulation which is used for the unstructured grid can be always used for the structured grid and one can come back to that and get that information. But for unstructured grids while the stencil used for the face gradient might not be decrease, the gradient across a face will still be based on the nodes which are connecting the faces. So, that essentially takes care of the gradient calculation and what we have looked at is that the gradients at different options and the using both compact stencil and both the extended stencil.

So, what we have concluded here is that talking about the basics of the CFD finite volume methods, essentially the discretization of the finite volume method and how you convert your governing equation to the discretatized system or the linear system and in this context we have discussed the diffusion equation. So, steady diffusion then unsteady diffusion so both on structured and unstructured case. So, that essentially concludes the content of this finite volume method 1, but now since we reach to the linear system so, we need to know certain properties before we have a solution for the linear system.

So, you will move to the next section and we will discuss few important properties of a linear system before we discuss about the linear solver and convection diffusion system in our second series like finite volume 2.

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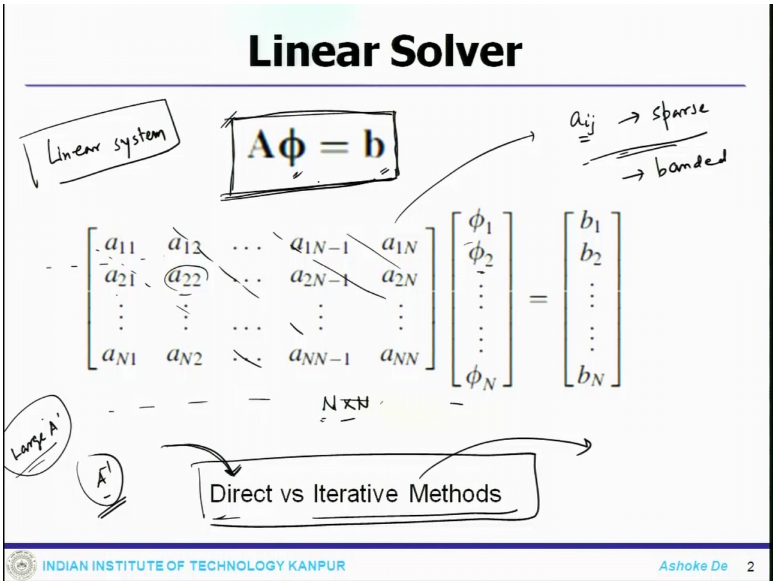


So, we will move to the next section of the properties of the linear system. So, now, once we are done with the gradient calculation essentially in the diffusion term discretization equation you can see that now you have obtained from your diffusion equation you get the matrix which is a order like this. So, you get the linear equation. So, this is my linear system. So, once I have discretized my equation system we obtain the linear equation in this kind of fashion.

So, this should be again true when you do the complete convection diffusion term or rather complete fluid flow problem equations, but now it is a time we should look at how you get an solution for that linear system. Now, have done the discretization for the diffusion system once you do the discretization you finally, get back that equation in terms of this linear system which will you can think about. Now, what are the process of getting the system solved? There are primarily two approaches one can adopt. So, one approach is essentially I have a matrix a variable to be found out and the right hand side vector.

So, what one can do, one can adopt a direct approach; that means, what you do in the direct approach you get directly the A inverse or one can solve iteratively. And secondly, also what you can see each row here actually they correspond to some sort of an element a i j. So, for second row second element means this a 22. So, any of this a ij this is nothing, but the measures of the strength of e because that is multiplied with second element on the variable. So, every row represents a equation defined over one element.

So, that is on that in the competition domain and the non-zero coefficient of these is going to get you the solution for that system. Now, what are the certain properties of this particular matrix? They are going to be sparse matrix, sparse matrix in the sense for some locations you might have the coefficients others are 0. So, that is what it is going to have and other hand you can have largely sparse; that means, you have a lot of 0 or less sparse; that means, you have less 0's. So, depending on the element or the coefficients actually this matrix could be of any nature and also sometimes when you look at the structural grid or when you discretize the equation system in the structure grid it can also get you the banded matrix.

So; that means, it may highly possible you can have some sort of a band like that or some sort of a band like that. So, that is also possible that you can get back some sort of a banded matrix in nature. But in reality or naturally most of the time the matrix what you get and that is a sparse system, now when you go to the direct approach how you can solve it basically you can get the direct inverse. So, now, that immediately gets you some sort of a problem when you have a large A then getting that inverse is computationally very very expensive.

You require large memory, you require heavily computing power. So, that is essentially for a large system one can think about its impractical. And when you talk about the physical problem these matrix is going to be some sort of a N by N and this n is the number of total cells count in your two-dimensional domain and when it goes to three dimensional you can think about the size. So, for a large system getting a direct method is absolutely impractical and that is why what people have done they have gone back to some sort of iterative process. That means, you consider this system, linear system and then you solve iteratively to get the linear system solved and through some sort of an minimization of the error.

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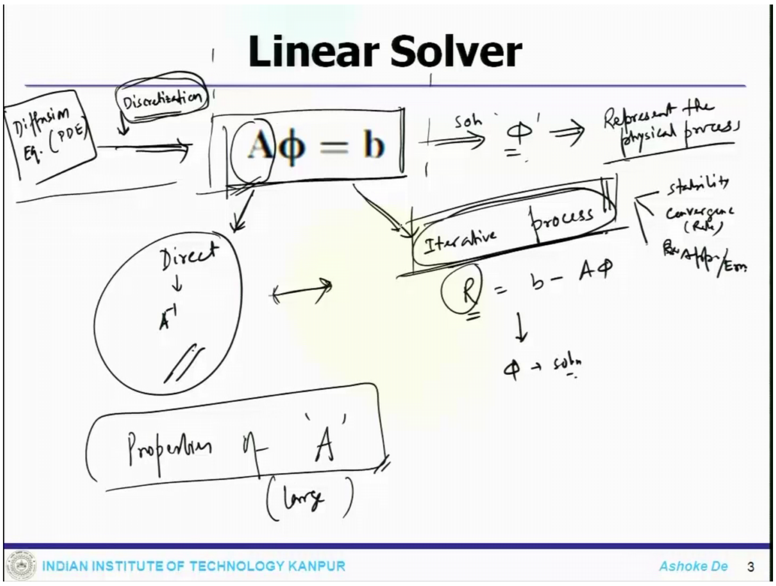


So, when you have this big linear system the one option as you said direct option where you get the direct inverse, other option is the iterative process and what you do in the iterative process that you some sort of a get some error function which is going to be b minus A phi and try to minimise this error and go close to the solution of phi. So, that is how you get it. So, this is heavily computationally expensive has serious limitations on large scale problem, but on the same time this guy is very handy for large scale problem or rather particle systems.

And one of the very much preferred method for linear solver in modern days CFD calculations where, the direct methods are can very much restricted to the small scale problem or academics problem. But large scale realistic problem or for community prefers to get iterative solvers and there are lot of methods which have been devised based on iterative methods how quickly one can get it. So, the important factors which will be again associated the stability then convergence. Then rate of convergence or rather convergence rate, then approximation or the error.

So, these are the things again one need to consider for looking at the iterative method. So, if you look at the picture you have a system basically you had the diffusion equation since, we have discussed the diffusion equation. So, that is nothing, but the PDE you have that you apply discretization and regarding the discretization whatever issues that can be associated like structured greet, unstructured greet, orthogonality, non-orthogonality, gradient computation, boundary condition everything includes in this block of discretization and that once we apply here that lead to the linear system.

Once you get a linear system you need a solution and solution only get you the variable phi. So, in the every note of the domain which you can get and that can represent the physical process so that is how it closes the loop. Now, we have done up to this much, now we are at this stage. So, we need to see how we can find a solution to the linear system; that means, A phi equals to be or A phi equals to b which can be a representative solution for a physical process. So, to do that you have couple of these approaches, but underline phenomena or the component which lies or becomes very important is the properties which are associated with these matrix or the large matrix.

So, one has to know certain properties of this linear system before moving ahead with any kind of these solver calculation because the solver calculations would get you an flavour of how to get the direct approach or the iterative process to get a solution for free. But that involves those assuming the properties of A which is the backbone of this calculation process. So, what we need to do next is the talking about some properties very quickly and once we do that we will move to the calculation of this direct solver.

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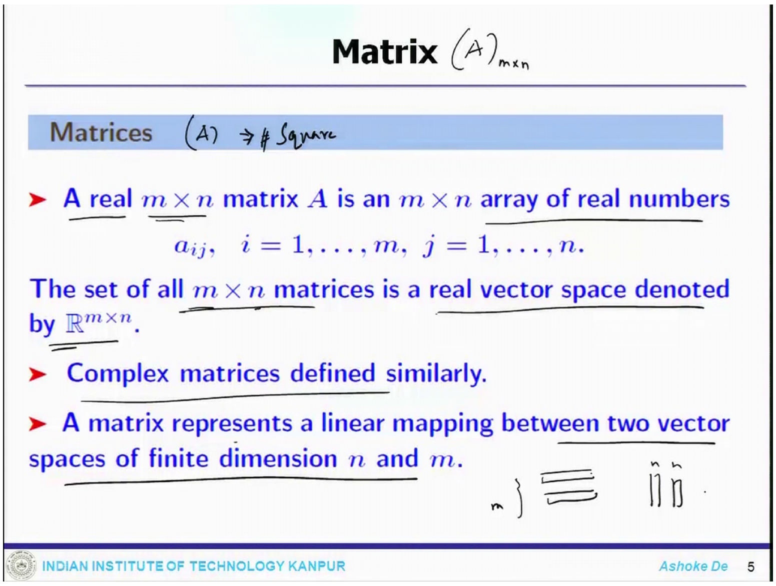


So, with that objective in mind we will look at that some properties of this matrices. So, essentially what we are talking here if this large matrix A. So, that is what is our point of concern and so the matrix is of order it is a real matrix because it cannot be imaginary then you cannot have solution. So, that is a first information that you have, it is a real matrix and in generic sense you could say it is a m by n system. So, what essentially it assume it is not a not a square matrix.

So, if again these are all generic synonyms if m equals to m then it will become square matrix, but when we do the discussion the m by n system that is what we will talk is certain array of real numbers that number one. So, the set of all these m by n matrices is a real vector space denoted by m by n.

So, now once you go back to this matrix it essentially all its row vector and column vector. So, there are m row vector, there are n column vectors and they are form some sort of a vector space denoted by this real expression. So, the complex matrices are defined separately and the matrix represents the linear mapping between two vector spaces of finite dimension n and m. So; that means, I have set of some row vectors these are m row vectors and I have certain n vectors which are column vectors.

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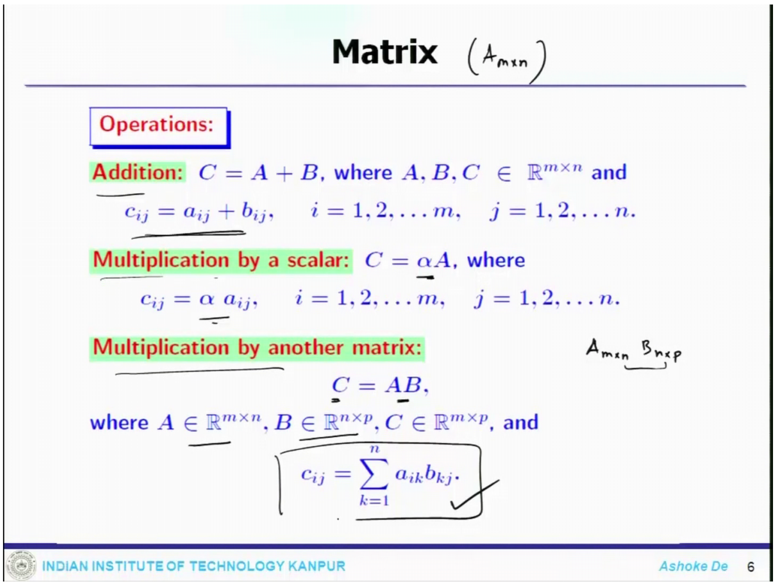


So, these are very important property operations if you look at the matrix operations. So, what we are talking about A m by n. So, the matrix operation you can have addition of the matrices.

So, the elements can be added which are very simple, if you have a 2 by 2 system you had the 2 by 2 system the corresponding elements can be added. You can multiplication by a scalar, if you multiply by a scalar each of this element is going to be multiplied by that factor alpha and then if you multiply by another matrix. So, the important property for the multiplication is that I am trying to get a matrix which is C A by multiplication of A and B. So, A is m by n. So, B has to be n p something so; that means, the A is m by n and B has to be n by p. So, these two dimension they need to be same then only the multiplication is possible.

And that means the number of columns and the number of rows in B; number of columns in A and number of rows in B they need to be equal then only you get the multiplication by this kind of approach.

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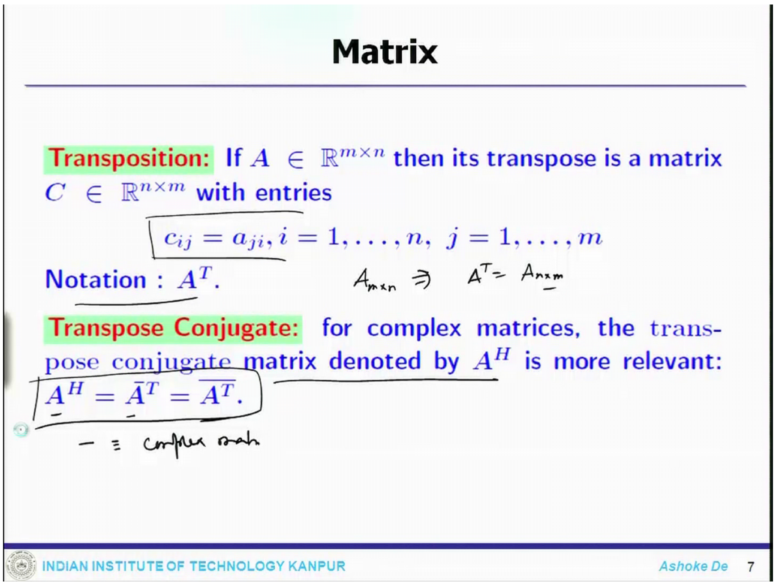


Transposition; transposition is the matrix how you write the transpose. So, if you have A R m by n matrix then the transpose of the matrix should be n by m. So, if a is m by n the transpose is A n by m that is how and you can symbolically write c ij is a ji transpose conjugate for complex matrices the transpose conjugate matrix denoted by A H or is the.

So, A H is A bar transpose, A bar means here the bar represents the complex matrix and that is the transpose bar is going to be the equivalent of same.

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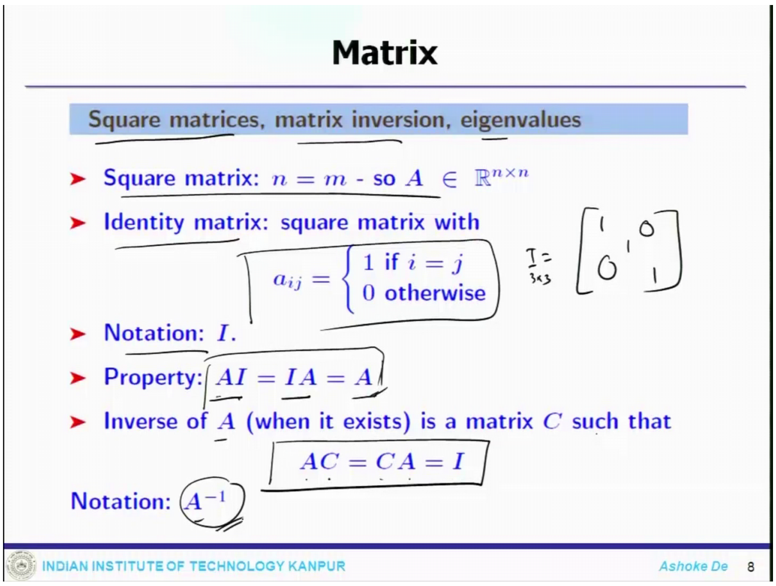

These are very important properties that you have for the matrix. Now you come square matrix that is one important square matrix as I already said m equals to n and becomes square matrix, matrix inversion. So basically, eigenvalues so identity matrix is if you have 2 by 2 system or 3 by 3 system the diagonal elements are 1 and otherwise all the elements are 0 that is the identity matrix for any.

So, this is 3 by 3 system, now similarly the if notation is I. So, property of AI identity matrix that, if you multiply to the any matrix whether in the right hand multiplication route or the left hand multiplication route you get back the A. So, if identity matrix actually does not change the property of the original matrix. So, that is one of the important information that one should keep again inverse of a when it exists is a matrix C such that if A multiplied with C and C multiplied with A it actually gives you back the identity matrix. So, the inverse is rotation and just we have said that for the direct method of the linear system solution you need to find out the inverse. So, you should know how inverse can be calculated.

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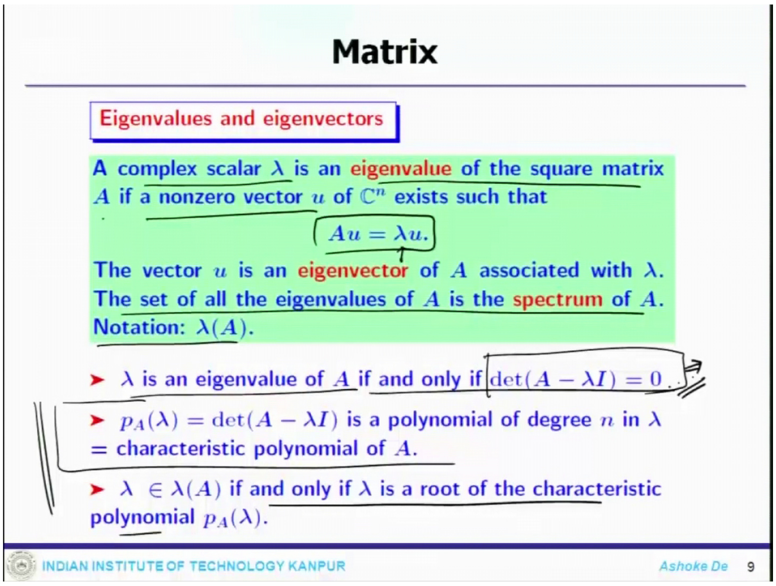


Now, what is eigenvalue? Eigenvalue is the complex scalar lambda is an eigenvalue of the square matrix A if a non 0 vector such that A u is lambda u and this lambda is going to be the eigenvalue.

So, the vector is the eigenvector and the associated values will be lambda and the set of all eigenvalues of a is the spectrum of a and one can write that eigenvalues of A by lambda A if lambda is a eigenvalue of A if and only if determinant is 0. So, this is the criteria that so that leads to a some characteristics polynomial and it depends on the size of the matrix if it is a 3 by 3 matrix and then it would be the characteristics polynomial of third order if it is 4 by 4 fourth order. So, it gives you the depending on the size of a that order polynomial and once you solve that polynomial the root of the polynomials are going to be the lambda.

So, this is what exactly its means the characteristic polynomial. Now, if lambda belongs to lambda A if and only if lambda is a root of the characteristics polynomial. This as I said when you get this equation the root of the characteristics polynomials are going to the representative lambda for this system.

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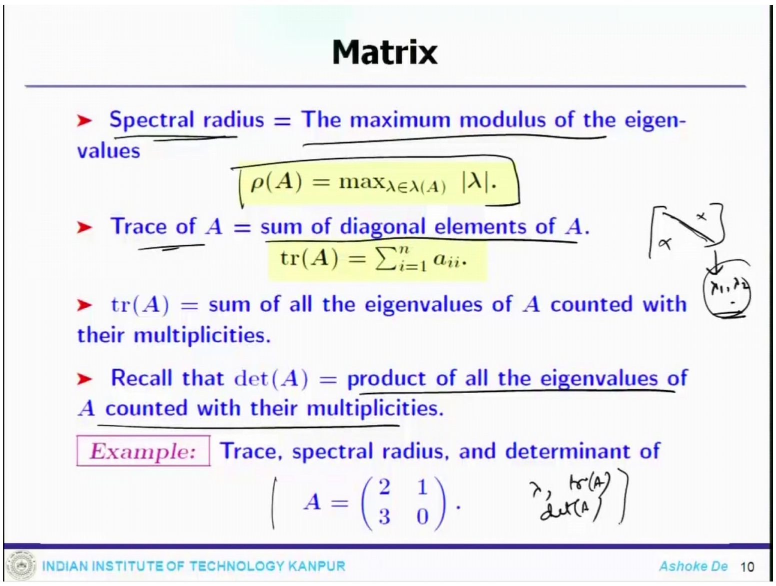


What is spectral radius this is again going to be an important information or the component that would be used while we will be talking about iterative process or iterative method for the linear system. It is the maximum eigenvalue or the mode of the maximum eigenvalues. So, it is denoted by rho; trace of A is written as the sum of the diagonal elements of A.

So, if I have A and some diagonal elements sitting there, these are some element. So, the trace is like that and the determinant of A would be the product of all the eigenvalues of a counted with their multiplicities so; that means, if I find out some lambda 1, lambda 2 and so on. The as long as they are distinct the multiplication or the product of those are going to be the determinant of A for example, let us say A is 2 1 3 0. Calculate that, you will find out what is lambda what is trace, what is determinant? So, all these you can find out from finding the lambda.

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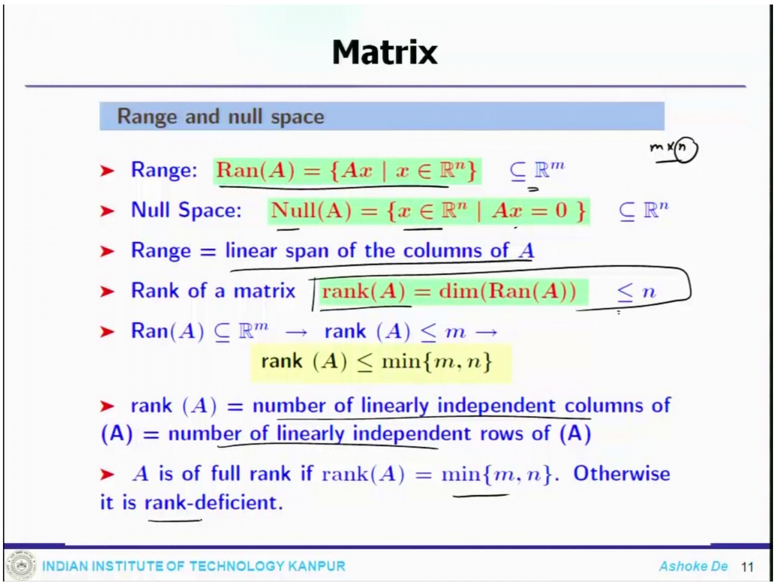


Then since, the rows and the columns of A they form some sort of a vector space it good to know what are those spaces and the range. Range would be belongs to R to the power n the null space is essentially a vector space belongs to R to the power n. So, which you will be solving this vector the null space vector by solving A x is equal to 0.

Since A has m by n system the null space belongs to n, then the range is the linear span of the columns of A rank of the matrix. The rank of the matrix will determine it is a number of independent column it has. So, if you have a three by three system and the three by 3 system if it has all 3 columns which are linearly independent the rank would be 3. So, that is why it is less than equal to 3. So, the maximum possible rank for a m by n system could be the n otherwise, it has to be always less than n.

So, it can be find out through the elimination process. Now, rank A is the number of as I said linearly independent column or linearly independent rows of A, If A is of full rank; that means, otherwise A is rank deficient.

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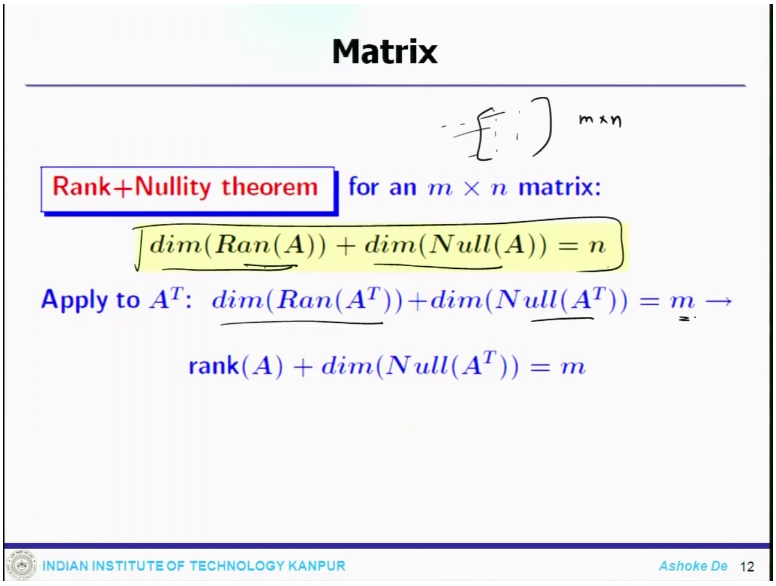


Now, you have a rank of A, the dimension of that and dimension of the null space should be n. So, this is a very important equation; that means, I have a m by n system and the number of independent columns are the going to be the rank. So, the dimension of that plus the dimension of null space so null space is essentially get you back the independent rows and that total is n. So, once you apply that or the dimension of the A transpose and null space of a transpose is m.

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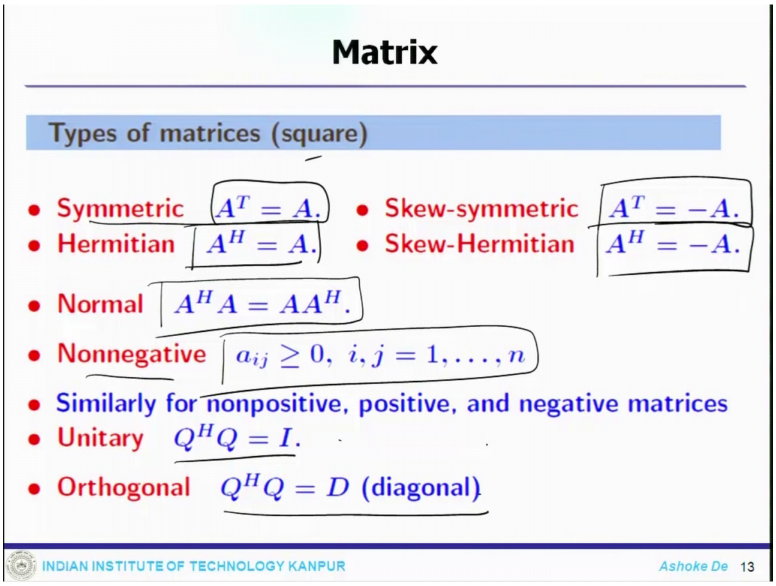

So, types of the matrices. So, it could be square matrices when n into m is equals to n symmetric matrices A transpose equals to A transpose, Hermitian matrices A H equals to A. Skew symmetric when A transpose equal to minus A, skew Hermitian A H is equals to minus A, normal A H A is AA H the matrix is non negative when its components are greater than 0; that means, they are not negative similarly unitary matrix, orthogonal matrix.

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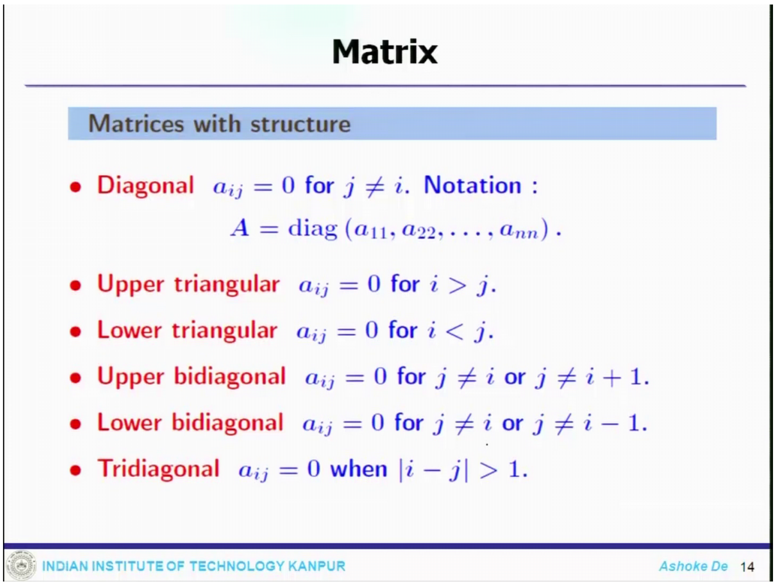


So, now, this unitary orthogonal matrix these are going to be important matrices which are now, another important thing is that matrix with the structures. So, these are the informations which are essentially going to be used and now the other properties that we are going to look at in the follow up lecture.

Thank you.