Instability and Transition of Fluid Flows

Prof. Tapan K. Sengupta
Department of Aerospace Engineering
Indian Institute of Technology, Kanpur

Module No. # 01
Lecture No. # 29

So, now, we have been discussing about spatial instability, and in the context, we were looking at the final details that within the spatial instability problem itself, is there a possibility that time dependence exists. And, that is what we were doing in this part of the course, where we are talking about spatio-temporal instability, and we were following the Bromwich contour integrals.

(Refer Slide Time: 00:53)

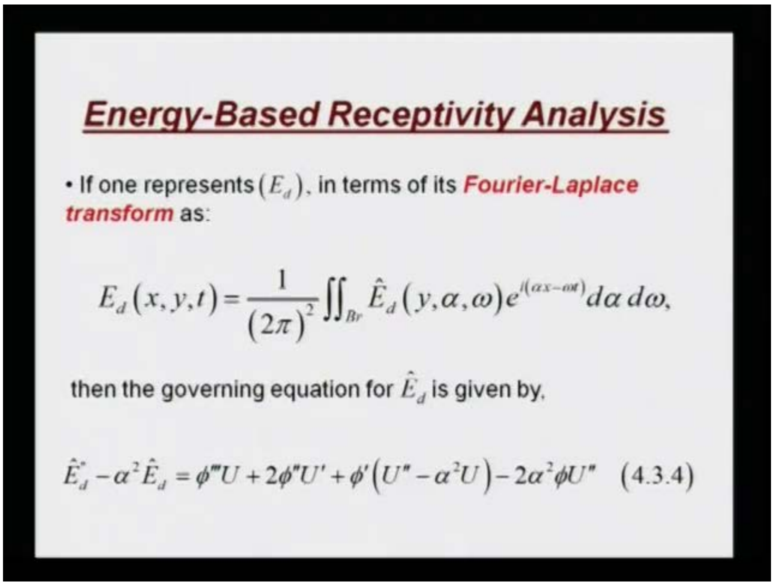
Energy-Based Receptivity Analysis

Table 4.1 Wave properties of the selected points

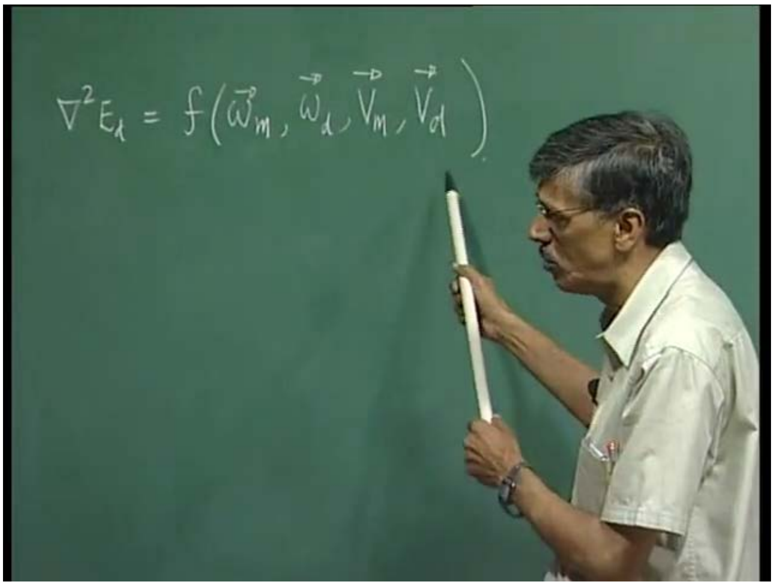
Mode	α_r	α_i	V_g	V_s	V_e
A1	0.279826	-0.007287	0.4202	0.42	0.42
A2	0.138037	0.109912	0.4174	...	...
A3	0.122020	0.173933	0.8534	...	...
B1	0.394003	0.010493	0.4267	0.352	0.352
B2	0.272870	0.167558	0.2912		
B3	0.189425	0.322635	0.1159		
C1	0.246666	0.013668	0.5026	0.50	0.50
D1	0.160767	0.001520	0.3908	0.33	0.33
D2	0.062141	0.069659	0.2762		

And, in the Bromwich contour integrals, we just looked at few representative points. We performed the Bromwich contour integral along two contours, in the wave number and the circular frequency plane, and we investigated four representative points; point A corresponded to an unstable scenario; point B corresponded to the same Reynolds number, but significantly higher circular frequency, whereas, point D corresponded to the same Reynolds number, but significantly lower frequency. And, C is a point, which is a sub critical point, that is, this is for a Reynolds number of 300; while the other three cases are for Reynolds number of 1000. And, the fact that point A is unstable is represented by one of the negative alpha i; whereas, rest of the modes, all seem to be positive. We talked about the group velocities. We also talked about the signal speed; signal speed is what Sommerfeld started talking about, and finally, we did talk about the energy propagation speed.

(Refer Slide Time: 02:24)

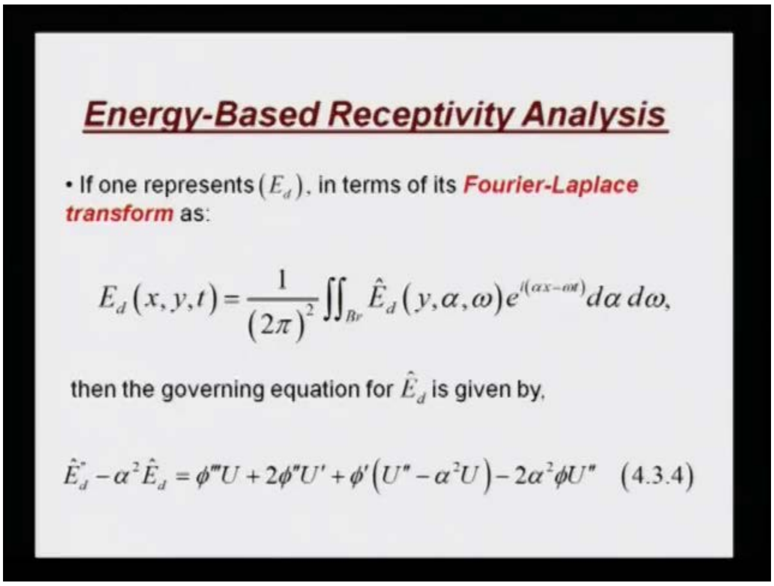


(Refer Slide Time: 02:41)

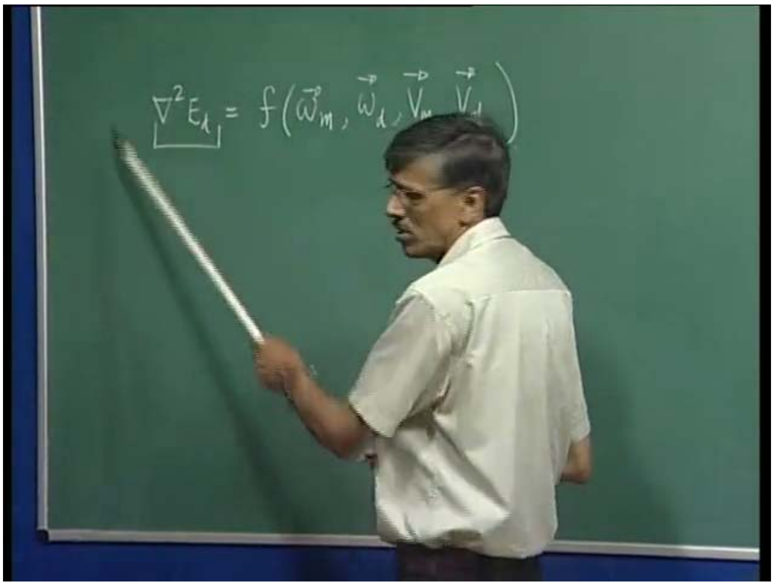


This was what was proposed by Brillouin. And, in this context, we started looking at energy based receptivity theory. Please do understand that I emphasize, that, this is a energy based receptivity analysis; this is not your stability analysis. Why did I say that, because your governing equation for the disturbance energy, that we have written here, in Fourier Laplace formalism, is governed by this equation. And, this equation came from here, del square E d, which is a sub-function of the mean vorticity, the disturbance vorticity, the mean velocity and the disturbance velocity.

(Refer Slide Time: 03:15)

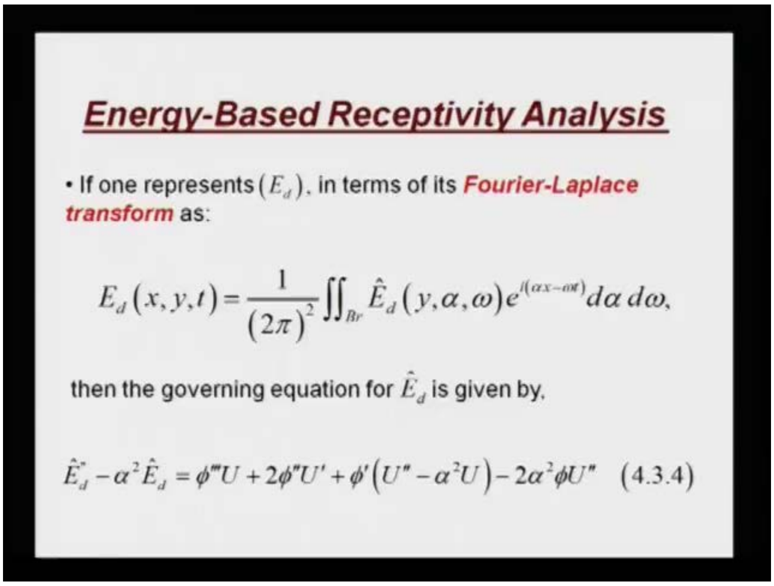


(Refer Slide Time: 03:27)



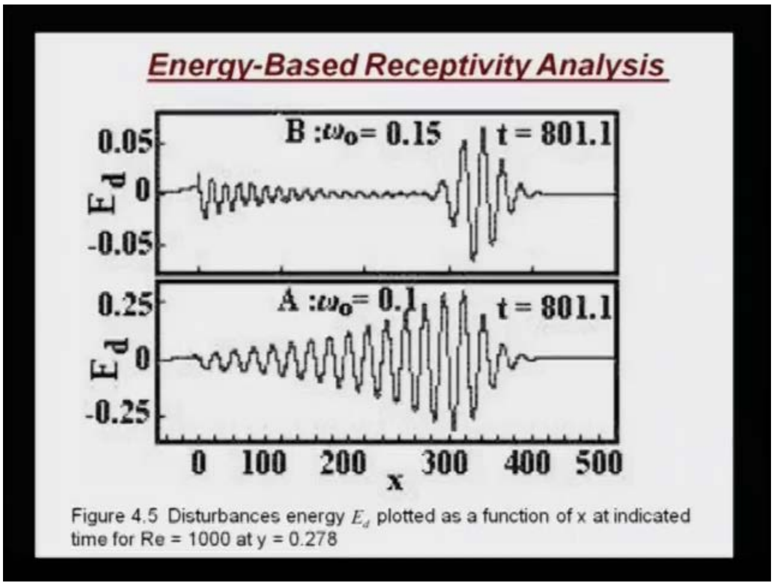
And, when you look at it, that this, there is a distinct forcing for this case, unlike the Orr-Sommerfeld equation. Orr-Sommerfeld equation is a homogeneous; this is a inhomogeneous equation. So, this is the forcing term. So, once you solve the Orr-Sommerfeld equation, you can see for yourself, what drives the energy. And, if you look at the corresponding homogenous equation, that is nothing, but the Laplacian. And, we know that Laplacian does not exhibit any instability, simply for the reason that this part does not have any time dependence.

(Refer Slide Time: 03:56)



If it does not have any time dependence, then, of course, you can talk of its instability. So, what instead, we are talking about the receptivity. So, if I prescribe some kind of a disturbance in the flow field, that will create the vorticity field, that will conspire with the mean flow to give this kind of a forcing on the energy. So, this is something, we must keep in mind, when we are talking about energy based approach that, this can at best, be a receptivity analysis; can be a stability analysis.

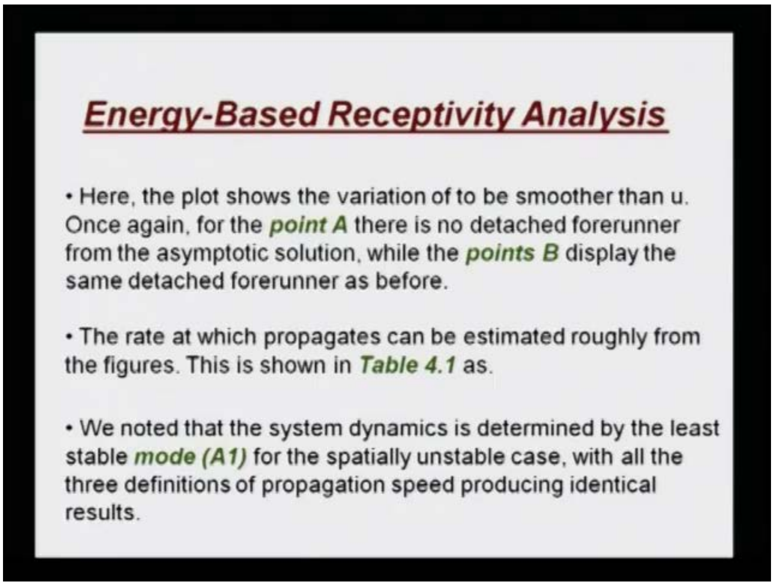
(Refer Slide Time: 04:14)



So, what we did was, we performed those integrals for those two points. The one below, is for the unstable frequency of omega naught equal to 0.1, for a Reynolds number of 1000. The result that you are seeing here is at a height of 0.278 and at a terminal time which was allowed by the string of data that we took, is given by this. And, what you notice here, is a sort of asymptotically growing solution, merges with the leading edge of the wave packet. So, here, the existence of the spatio-temporal wave front and the asymptotic part of the solution are not distinct. Well, that is distinct for the stable case; for the stable case, what we see of course, the asymptotic part is decaying; but the spatio-temporal part, which actually grows in space and time, continues to be there. And, you can see them very distinctly. If we would have aligned these two figures properly, we would have seen that, they would have matched and they would have been the same.

And, this prompted us to comment in the last class that, this is perhaps very strong function of Reynolds number, and these two are for the same Reynolds number, but for different omega naught. But the spatio-temporal solution is, being a function of Reynolds number alone, and these two being same, they have a identical structure. So, this is what we commented, but at the same time, I also made the observation that, this needs further probing.

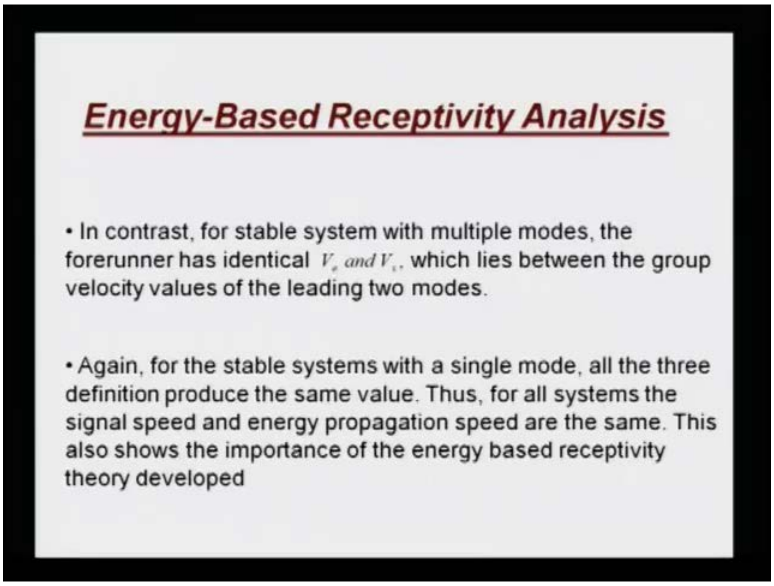
(Refer Slide Time: 05:53)



So, basically, in summarizing it, we have basically talking about the same thing, that when we are looking at the E d variation with x, we see that, that is much more smoother than u. The reason is, E has that V square by 2. And, well, I mean, this is not a quite a carefully made statement. What we are seeing, that we do not have a detached forerunner; we are not saying that there are no forerunners. So, please do understand that there is a detached forerunner; while for the point B, you can simply see the distinct detached forerunner. The rate, at which this asymptotic part, as well as the forerunner propagates, can be roughly estimated from the figure itself. We have the visual signature; from there, we can calculate this, and this was what was shown in the last column of the table; we just opened up [no audio from 7:04 to 7:14] mode 1, that is that unstable mode.

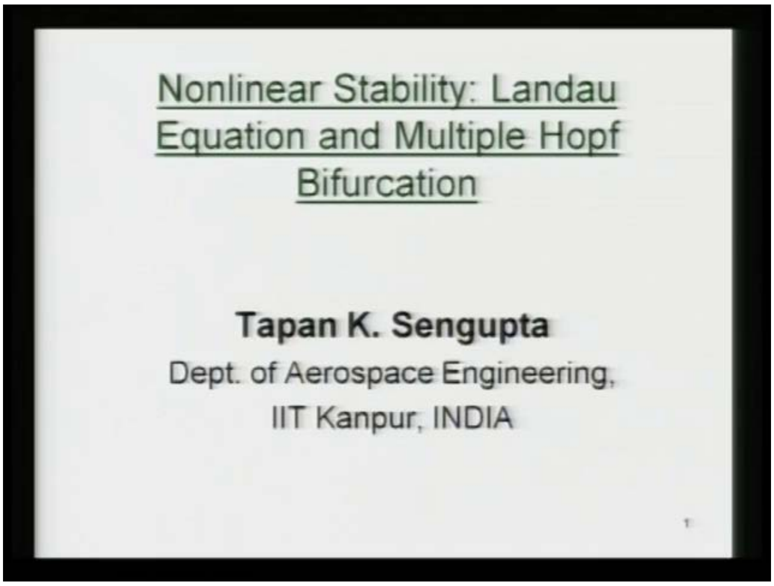
And, it just so happens that if you do a Fourier analysis of the forerunner, that corresponds to the second mode, in terms of the wave number alone; because, you see, the second mode is a stable mode; however, we, what we see in our calculation, we have calculated all the way up to 800, and this spatio-temporal wave front kept on increasing; it is not a decaying thing, like the second mode.

(Refer Slide Time: 08:04)



So, even if I say, that in terms of wave number, it matches the second mode, it could be just pure coincidence. So, it is not a causal factor. And, what we found that when we are looking at stable system, like the point B or D, what we find, those have multiple modes, and the forerunner has identical, the energy propagation speed, as well as the signal speeds. And that, lies between the group velocity, between the values of the leading modes. For the stable system, we have a single mode alone; we do not see the forerunner. And, that case also, once again, I emphasize, was for a Re equal to 300. So, maybe it is such a strong function of Reynolds number, that for I equal to 300, we do not have a forerunner.

(Refer Slide Time: 09:02)

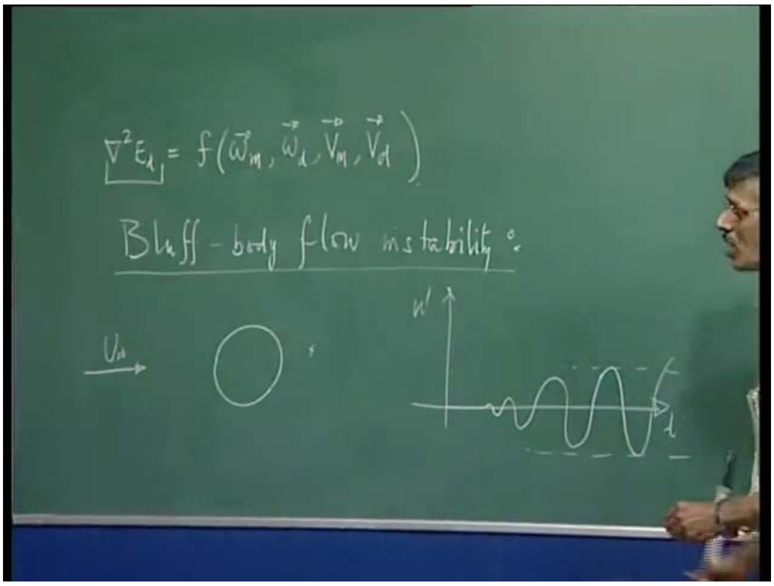


So, there is a distinct need for performing this kind of analysis for different Reynolds number, and see, how this forerunner behaves with time. Let me now, go over, and talk about a new thing. What is the new thing that we are talking about? We are going to talk about, completely a different aspect of instability. Now, there are two things that I want to highlight here; number one, that we are looking at here is, here, we are not any more talking about spatial instability. We are talking about systems, which display temporal instability. And, one such an instability is there. This is going to be distinctly different from spatial instability. In the spatial instability what happens? The disturbance washes off; it convects away. And, if we do not take a very very large domain, we do not get to see the disturbances growing very much. That is more, a sort of a limitation of our computational domain. If we probably would take much longer domain, then, we could see that it is growing too much longer time.

Like, exactly what we just now finished our discussion on that, if we instead of talking 512 points in the omega plane, if we would have taken, let us say, 8 times or 16 times more number of point, we could go over for much longer time. And then, if we also take a larger x region, we would see that the asymptotic growing part will keep on growing, because it is a linear theory. Linear theory does not tell you where to stop. It just simply tells you, it is unstable; that is about it. So, that is the second part, that when we have a, such an unstable system, then, what does non-linearity do? We clearly understand, one thing I suppose, you will agree with me full-heartedly, when I say that the disturbances cannot grow completely unbounded. Why do I say that, because the energy for the disturbance comes from the mean flow, mean flow has a finite energy.

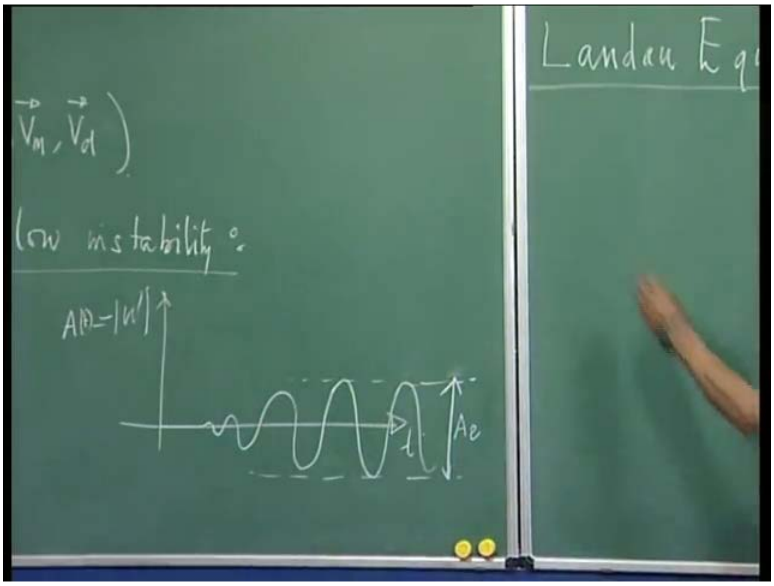
So, disturbance quantities cannot grow unbounded. In fact, the very phenomena of transition from laminar to turbulent flow, tells you something about moderation of non-linearity coming into picture. The linear theory or the instability theory says the disturbance grows. Where does it stop? That is the non-linearity, comes into picture there. So, in the second part, what we are talking about, we are talking about temporal instability, and we are talking about a finite domain. And, if such a system is linearly unstable, what am I going to see? With time, it is going to grow forever? And, I am doing real time plane calculation, not in the frequency plane, not by a Bromwich contour integral; suppose, I keep on doing for large time, what will happen? It will eventually block. Computationally though, for various systems that have been studied, which display temporal instability, we do not see such a thing; and one of the best example for this case is, a flow past a circular cylinder, a bluff body flow.

(Refer Slide Time: 12:27)



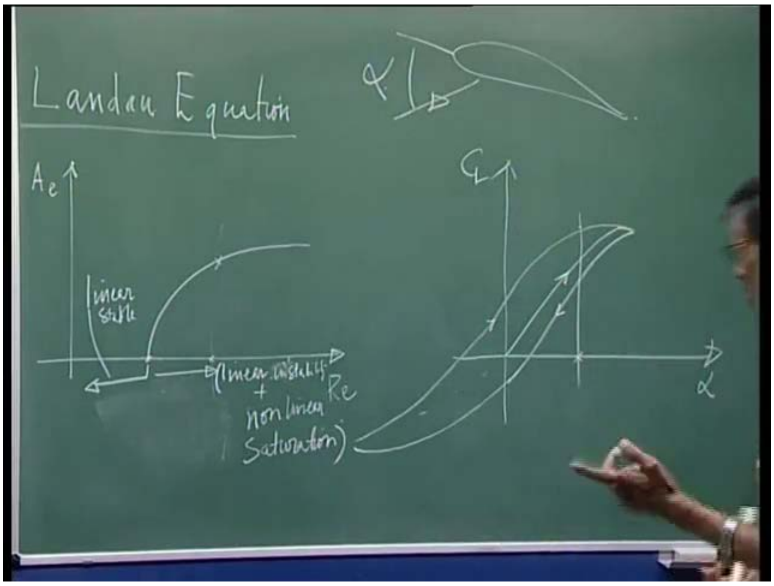
So, we are basically, next talking about, bluff body flow instability. If we are talking about this, our people tend to think that, this is entirely a different phenomena altogether, does not belong to stability study; but they try to always, approach this problem from the point of view of solving the full non-linear equations, so, Navier-Stokes equation. What do we see? Let us, first of all, identify what we see, and then, we should go about understanding what is going on. Let us say, we have a uniform flow, approaching this bluff body. Then, if I focus my attention on a particular point on the wake, what am I going to see? I am going to see something like this. If I plot the disturbance velocity, what I would see that initially it remains, then, it slowly builds up. And, once it builds up, it indicates some kind of a growth in time. When it shows the growth in time, the natural question that arises in our mind that - will this growth be forever? But what we notice, that it does not happen so; it actually, saturates.

(Refer Slide Time: 14:40)



And, this is where our prism discussion start. We are talking about effect of non-linearity, and this effect of non-linearity was studied initially by Landau, the famous Russian physicist. And, he produced an equation, which is called the Landau equation, which will tell you about the amplitude of the growth of this disturbance, and that kind of, tells us that why and how this growing amplitude saturates. So, this is what we are talking about. This is related to another phenomena, which is often studied in mathematical physics, namely the phenomena of bifurcation. Bifurcation means what? Now, this is also, where some amount of misunderstanding prevails. Supposedly, I call this modulus here, as A; then, what we are seeing that, this A is a function of time.

(Refer Slide Time: 16:10)



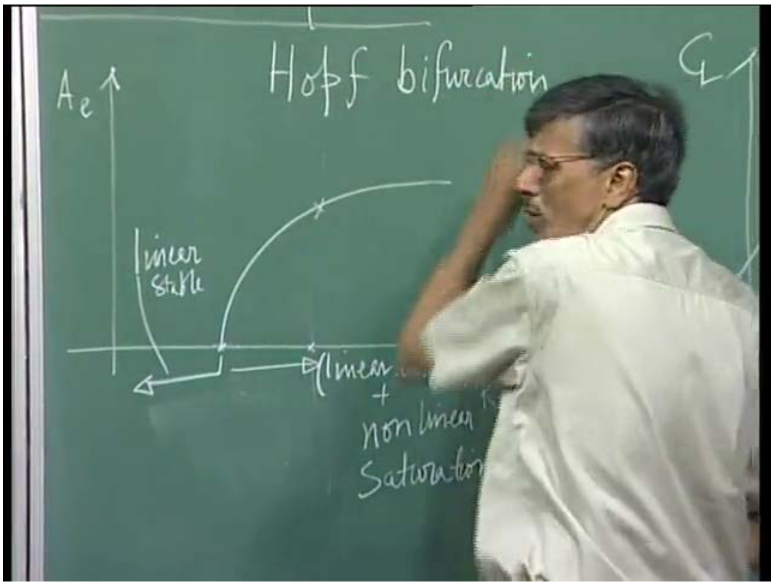
But because of the mediation of the nonlinearity, I get a saturation amplitude. This, I call it as Ae, equilibrium amplitude. Now, what I could do is, I could plot this equilibrium amplitude versus Re. I could do it experimentally, I could do it theoretically or computationally, and I can measure it, Ae as a function of Re. What I would see, that up to some Reynolds number, this equilibrium amplitude is 0. What does it mean? This system is linearly stable. So, this kind of instability does not arise for those Reynolds number. Then, after that, slowly, this instability starts. There is onset of instability, and with the increase of Reynolds number, this equilibrium amplitude keeps increasing. And, Landau proposed an equation, which shows that, this equilibrium amplitude goes like this. So, it is like a parabolic variation, Ae versus Re. And, this point, is a point, where we have the onset of instability. So, what we are saying then, on this side, we have a linear stability; then, if I perform a stability analysis, linearly, it is stable.

On this side, what I find, linear instability plus a non-linear saturation. This is the part that we need to understand. So, the point at which the instability first appears, what does it tell us, that if I am looking at an arbitrary Reynolds number, then, there are two possible solutions. What do these two solutions imply? This is what we are seeing there; this is what we have plotted. What does this imply? This implies there is no growth. How can it happen? Since we have studied so much on receptivity, you can give me an answer by saying, yes it can happen. Suppose, I do not have the corresponding input, I will not see the output. So, it is basically, this, that I can get this solution or that solution so; that means, the system from this point on bifurcates. It can have two solutions; one undisturbed solution; another is a disturbed solution. So, this point, at which this thing starts, is what is called as point of bifurcation. So, system bifurcates. In fact, you know, most of you have some exposure to various other aspects of fluid dynamical flow including, let us say, aerodynamics.

Here, if you realize that if I am talking about a flow past an aero fan, which is at an angle alpha, then, what I can do is, if I increase this angle of attack and measure the lift coefficient, I see that it goes like this; then, it has a stall; but now, suppose, I reduce the angle of attack, what happens? It does not follow this; it comes on this. And, you know that if I go on the other side, I could have a similar scenario, but then, when I increase it, I again may not go along this. So, I could get this kind of a picture. What is it called, of course, you, all of you know, it is called hysteresis. So, there are many examples of physical system, which shows hysteresis, which tells you what? That for a given angle of attack, you can have three possible solutions; and which solution you belong to, this is dictated upon the rate at which your angle of attack has changed. If I started from angle of attack 0, I would be on the middle branch; but if I would have come to the same angle of attack, coming from this tall angle, then, I will be here, and if I go back and again, retrace the cycle, and now again, I am on the increase, then, I will be here.

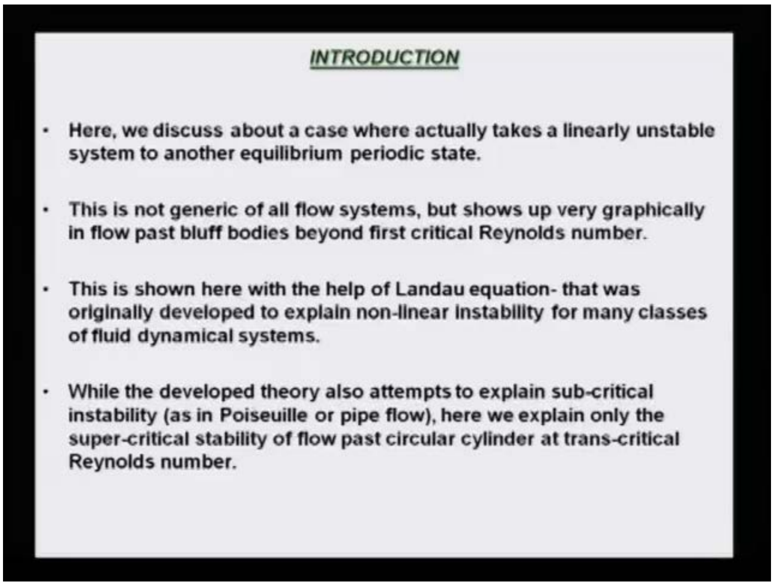
So, this is basically three possible solutions, and here, we are talking about two possible solutions. Well, this is a clear example of flow phenomena, which tells you the dependence of the flow on the time history, is it not. Where we have started, how we are going through that state, this is what dictates, whether I am going to on this branch or this branch or that branch. Here, the thing is slightly different. What it says, that if I am working on this Reynolds number, then, of course, I can have this unstable solution, saturated solution, disturbed solution, but I can also have this, depending on whether I have the corresponding input to the system or not.

(Refer Slide Time: 22:58)

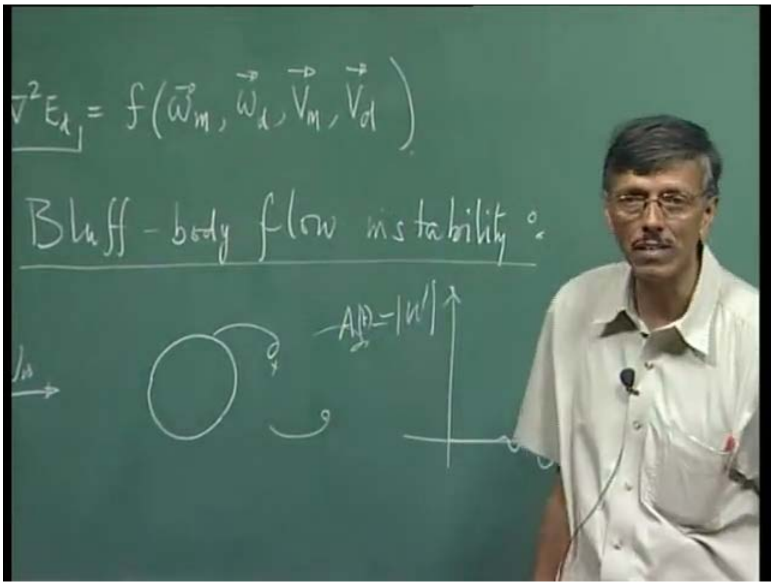


So, please do understand that, this is what it is. When this breaks up, the solution goes up like this, it is at a quadrature; this is what is called as Hopf bifurcation. So, that is your Hopf bifurcation. There are other kinds of bifurcation, like Pitchfork bifurcation etcetera. We will not talk about them. We will keep our attention focused on bluff body flow instability and then, we will see, where we do go from here.

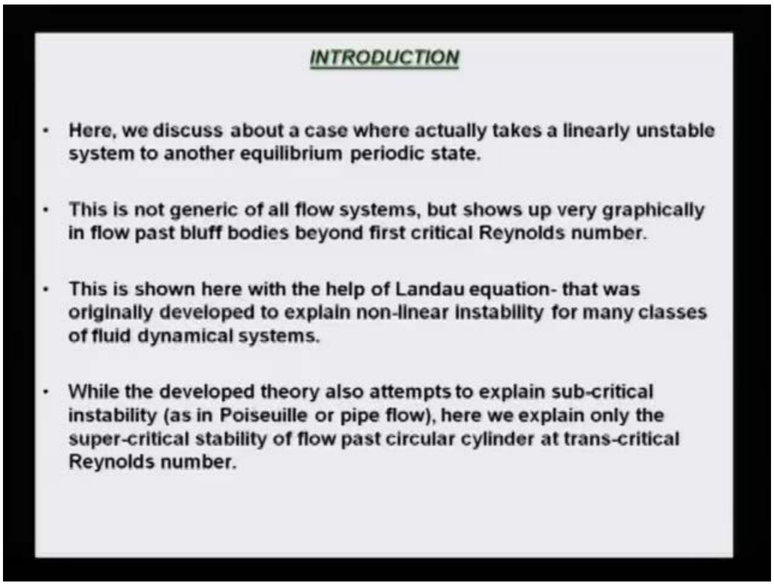
(Refer Slide Time: 23:31)



(Refer Slide Time: 23:58)



(Refer Slide Time: 24:09)

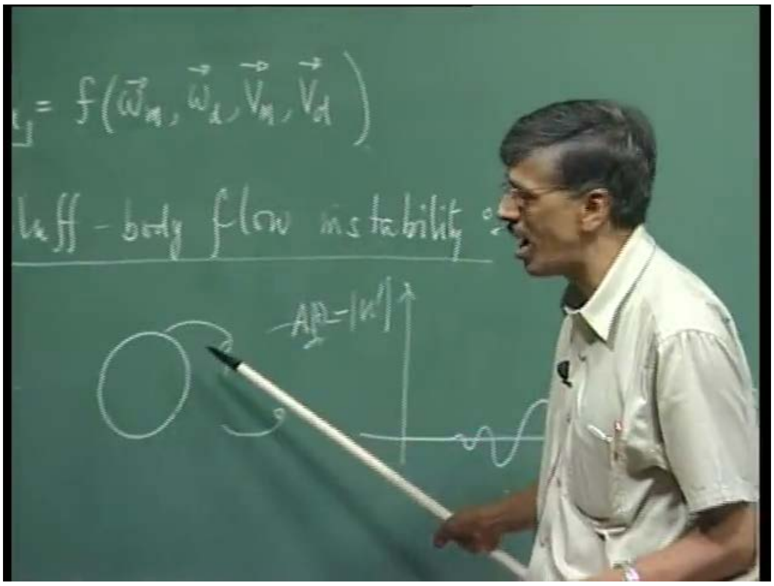


INTRODUCTION

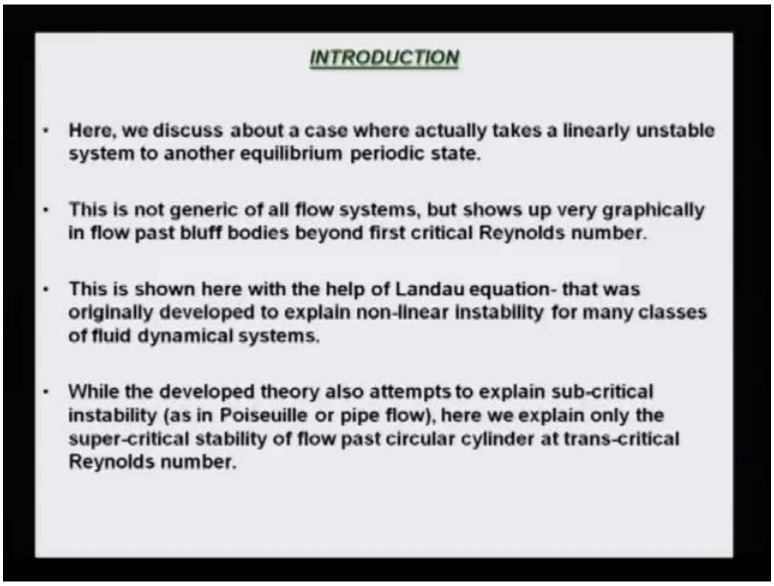
- Here, we discuss about a case where actually takes a linearly unstable system to another equilibrium periodic state.
- This is not generic of all flow systems, but shows up very graphically in flow past bluff bodies beyond first critical Reynolds number.
- This is shown here with the help of Landau equation- that was originally developed to explain non-linear instability for many classes of fluid dynamical systems.
- While the developed theory also attempts to explain sub-critical instability (as in Poiseuille or pipe flow), here we explain only the super-critical stability of flow past circular cylinder at trans-critical Reynolds number.

So, basically, when we are talking about non-linearity, effect of non-linearity and going to another saturation amplitude, etcetera, what we are talking about, we are actually talking about a linearly unstable system that goes to another equilibrium state. That is what happens here, that we do get this vortex shedding pattern, Karman vortex trait; that is a basically, that the next equilibrium periodic state. An oscillating pendulum, is also an example of equilibrium stage, and that is also a consequence of what was your original equilibrium state for a pendulum, to remain vertical in one place; you have disturbed it; then, it keeps going back and forth, and there is a tradeoff between the cause and response. You see, what happens, we see a basically, a disturbing force that is, the kinetic energy and potential energy does a perfect balancing act, and you get a constant amplitude oscillation, that is your pendulum.

(Refer Slide Time: 25:03)

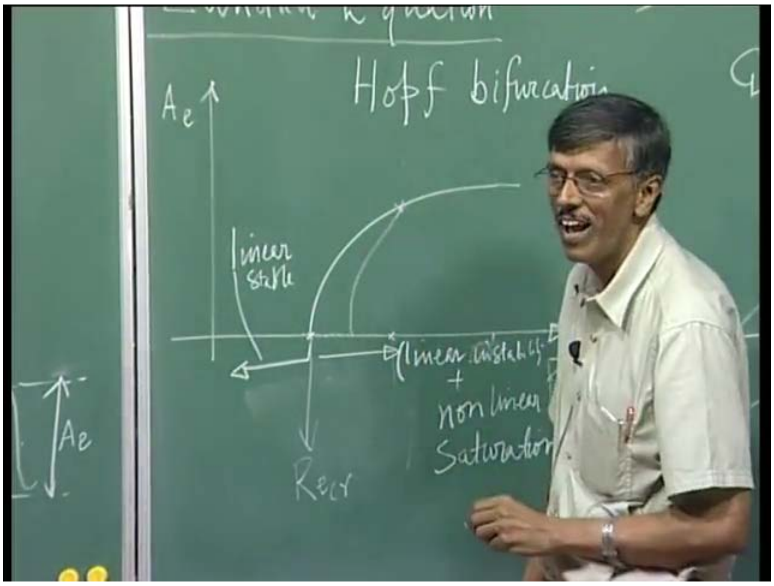


(Refer Slide Time: 25:31)



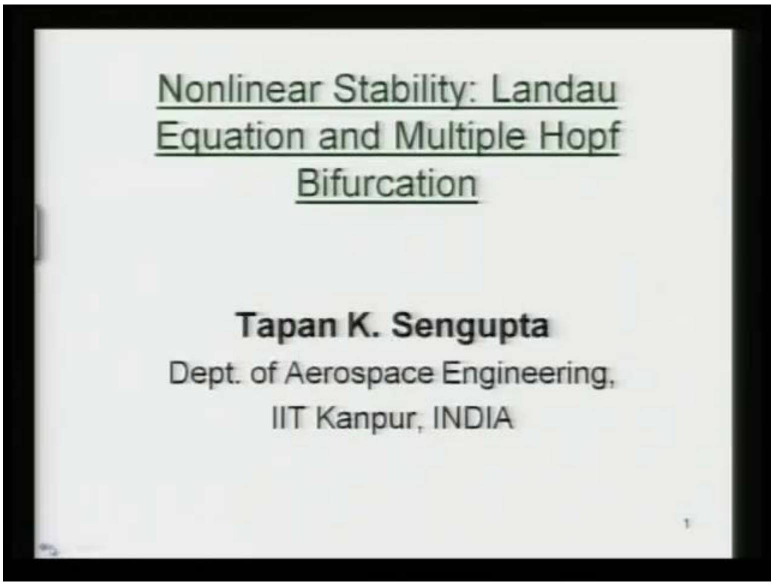
It is also the same thing here, vortex shedding behind a circular cylinder. It is like a fluid dynamical pendulum. Do you see the connection, that here, instability starts off, which we have in the linear mechanism; then, the non-linearity starts coming into play; that provides a kind of a saturation to it. We will see it, we will shortly come to the Landau equation, and there is a perfect balance. And, you come to this from a one linearly unstable state to another equilibrium periodic state.

(Refer Slide Time: 25:57)



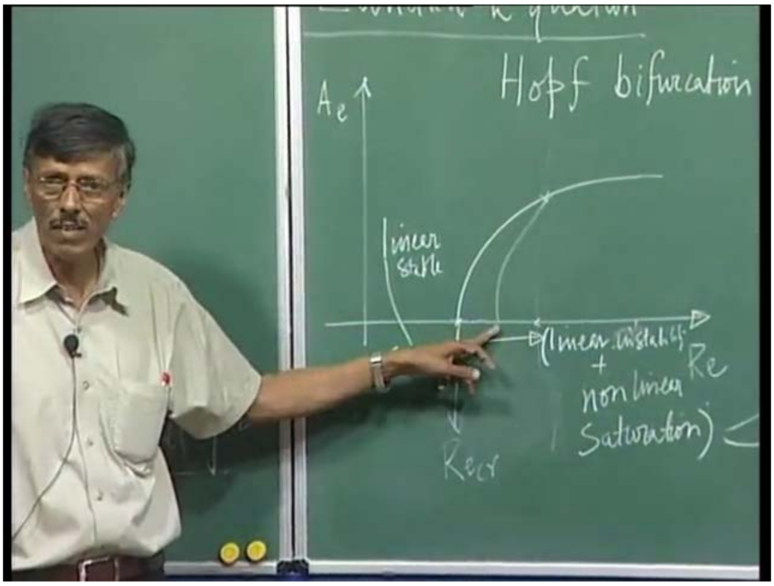
This is, of course, not generic for all flows, but it shows up very graphically in flow past bluff bodies, beyond the first critical Reynolds number. Now, this is the first critical Reynolds number, and I am, well, we have actually, by now, when this visuals were created two years ago, we were in the process of working out, that today we can very confidently say, the work is done, and we have shown that there is no such thing as a first critical Reynolds number. This again, is built into this picture, that whether I can remain here or I can remain there.

(Refer Slide Time: 27:14)



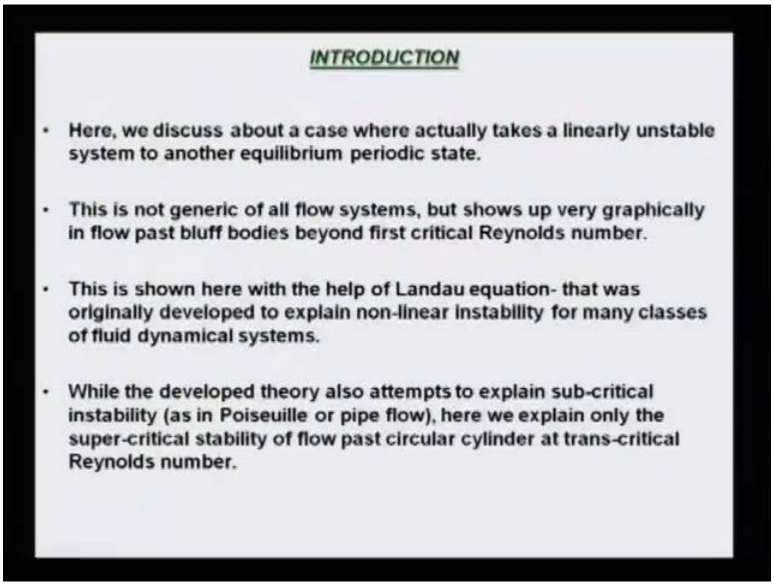
So, I will get this bifurcation, when I have that class of disturbance. Then, I could actually, consider a fluid dynamical system, where, this first bifurcation does not take place. If this is bypassed, then, I can continue to be here. And then, I can actually go to the second bifurcation. In fact, that is what we talked about in the title itself. When we talked about this a couple of years ago, people were not very sure, what we are talking about; what is this multiple Hopf bifurcation; people were only talked about Hopf bifurcation; and here we come, and we say, look, there need not necessarily be a definitive single bifurcation.

(Refer Slide Time: 27:37)



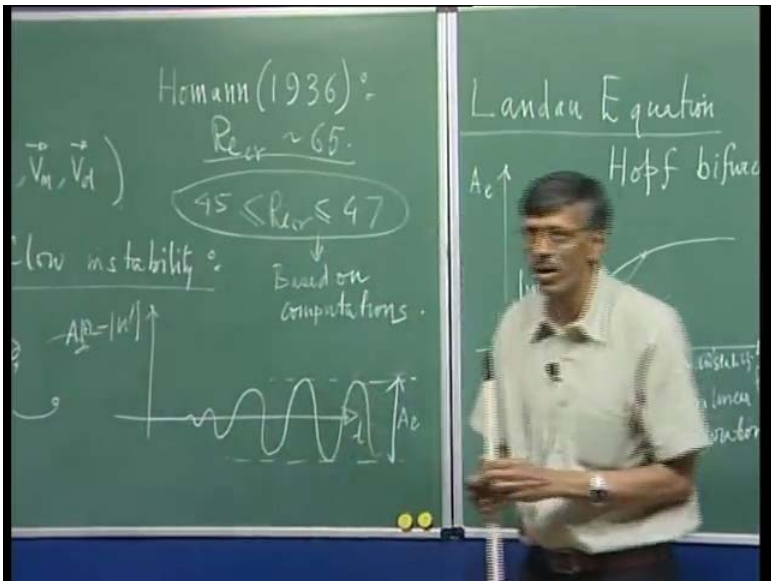
Because, that is what this bifurcation diagram clearly shows. I can bypass this first critical point, but then, there could be another class of disturbance, which triggers a second bifurcation, which can trigger a third bifurcation. And, let me just tell you, what you are going to see in advance, in this course that we have now established a scenario, where you would see multiple Hopf bifurcation, not only for this external bluff body flows, we have also shown it for internal flows. Wherever we have vertical flows, we have shown this; we have made a claim in one of our very recent work paper, which is in the press; there is a universality of this picture.

(Refer Slide Time: 28:39)



So, keep that in mind and then, we will go along, and we will see, how the drama enfolds. So, what we are saying that for flow past a bluff body, we can very graphically see this. Whenever we cross the first critical Reynolds number, and if there is sufficient disturbance in the back ground, we are going to see this. There is no doubt about it. But there was this experiment, done by one of students of Ludwig Prandtl, Fritz Homann. He did this experiment; he was very clever, far ahead of his time. What he did, he created a re-circulating tunnel, like what we have, the wind tunnels or water tunnels, people will talk of. Now, Prandtl probably suggested to him or they talked together and came out with the idea that they will build a tunnel with a completely different working medium. They used lubricating oil, which has a very very high viscosity. And then, in that re-circulating tunnel, they put in a circular cylinder and did the experiment and their experiment showed that there is no vortex shedding upto Re equal to 65.

(Refer Slide Time: 30:02)

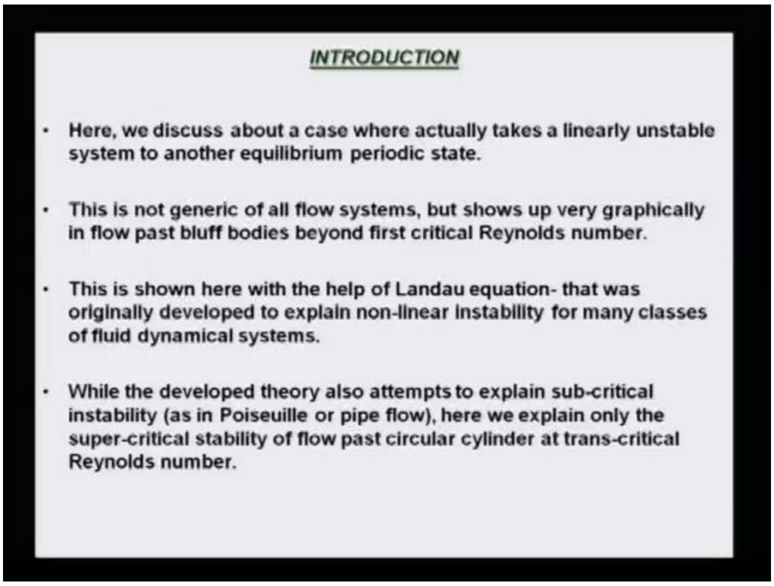


So, Homann's result showed a Re critical close to 65. Now, if you look at the literature of 80s and 90s, there was a nice bandwagon. Everybody said Re critical was 45 to 47 and people said this based on computation. This is another example of abuse of computers; people ganged up together, and made a lot of hue and cry, saying that flow past a cylinder, critical Reynolds number has to be between 45 and 47. And, here was this result, though it was not published per se, but it was a thesis, and people did know the result.

Because, those visualization results are in Schlichting's book, are also in White's book. So, two of the best known fluid dynamics book featured those results, but nobody was willing to reconcile between these two. Experimentally, you could go as high as Re equal to 65, and have no vortex shedding; whereas, here was another set of people, who were determined to say that, there is a universality, and this critical Reynolds number is about 45 to 47. Like, now, where is the truth; truth is of course, covered, in the sense, various other experimentalist haves come out with different estimates of Re critical. For example, if you look at the book by Landau and Lifshitz, they did not show any documented results, but they claim that the flow past a circular cylinder that becomes unstable for a Reynolds number as low as 30 itself. Batchelor's book also talks about, which also contains Homann's result, Batchelor's book also talks about, critical Reynolds number, somewhere around 40. And, there are any number of people who have done experiments, they talk about critical Reynolds number of anywhere between 40 and 55 and so on and so forth.

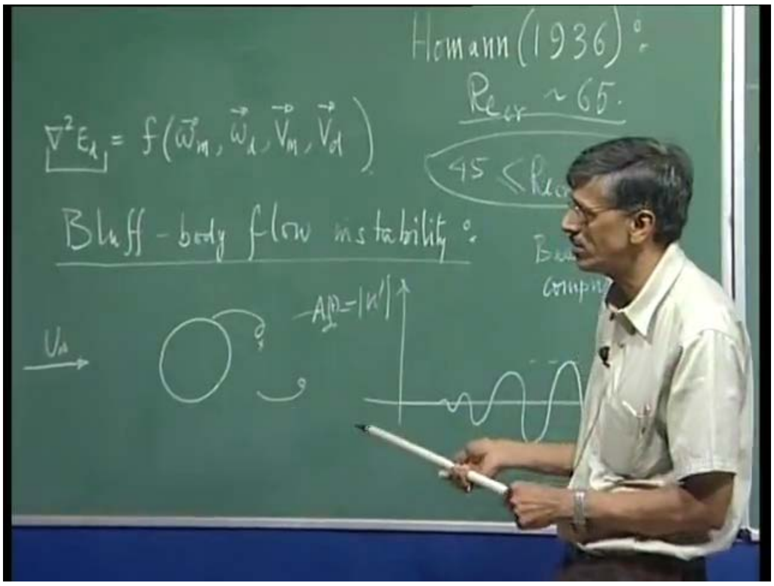
But no one actually, goes this high. And, I explained to you, why this high value was achieved. If you are working with a media of highly viscous fluid, then, what happens is that if there are any back ground disturbances, they are damped, because of viscosity of the medium. So, that was the unique thing, and it was only in paper, which we published in the last couple of years; we have shown the connection, what could be the reason. One thing is for sure, existence of the bifurcation diagram very clearly shows the role of receptivity; we cannot escape it. If we do not have the input disturbance, we are not going to see the response. And, that is what all this authors, all this experimentalists are talking about different Re critical, from the same point of view. And, to talk about an universal critical Reynolds number, in the range of 45 to 47, is a bit farfetched.

(Refer Slide Time: 34:31)



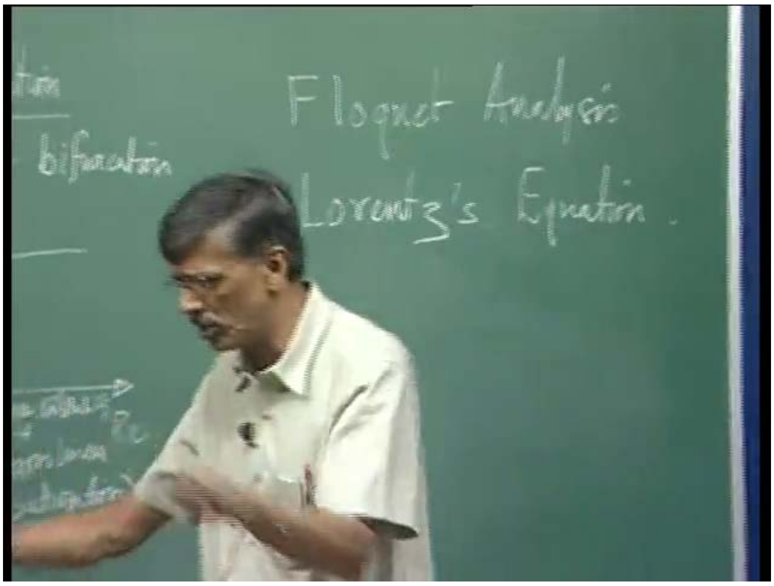
So, what we are going to do, now, we are now going to see, what really happens in this flow, and what was this Landau equation that we are talking about? Landau equation was conjectured by Landau. He did not actually give Reynolds steps in his work, but how did he come to that equation. Later on, this British mathematician JT Stuart, he did a very good bit of analysis with the help of Watson’s result. And, he showed that Landau’s equation actually comes about from the instability equation itself; the non-linearity comes from a self-interaction term, and that was a very good reason for showing this validity of Landau's equation. So, Landau equation, although was written in a heuristic manner in 1940s, it took another 20 years, before it was formally shown, how it is to be. So, that is why, today we call that equation as Landau-Stuart equation.

(Refer Slide Time: 36:07)



See, the, he showed that we are addressing (( )), is one of instability, but when Landau proposed his equation, he was more interested in explaining how turbulence comes about. What is the scenario he was talking about, that you have a primarily unstable flow, that leads to temporal growth of disturbances; non-linearity comes into play and you get into another periodic state; that is your first bullet. Now, once you have arrived at an altered equilibrium state, that is also susceptible to disturbances. So, what happens? That could suffer an instability, that would be a secondary instability. And, that could lead to potentially, another equilibrium state, that can suffer another instability and so on and so forth. So, Landau's view point in proposing this equation was that we are talking about such successive bifurcations and infinite number of such bifurcations or a very large number of such bifurcations, which leads a flow from laminar to turbulent state.

(Refer Slide Time: 38:01)

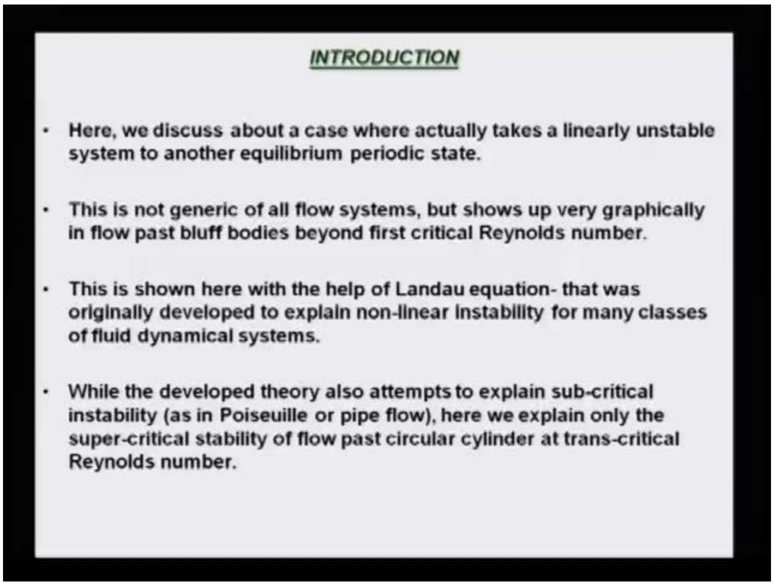


So, he understood the turbulent state has finite energy. So, it has to end in some equilibrium state. And, you also realize that what is going to happen. If we are talking about instability of an equilibrium state, the primary flow, the first instability, could have been on a steady flow. After the primary instability has occurred, what is the flow situation here? This equilibrium state is time periodic. So, a mathematical theory was developed quite earlier on; it is called Floquet analysis, Floquet theory. In Floquet theory, what you do, you study the instability of a equilibrium state, which is periodic. So, in a sense, what Landau was basically suggesting, is a kind of a primary instability followed by instability of periodic states; I am going from one periodic state to multiple periodic state and instability of multiple periodic state and so on and so forth, to turbulence. So, that was his scenario of turbulence. This is just a matter of record, it is bit of a story telling here that, in the 60s, people did come across, what was called as chaos theory, which showed that some of these flow phenomenon that we usually study, corresponds to very high Reynolds number.

And, these high Reynolds number flows are susceptible to small disturbances. And, some of the mathematicians came out and talked about, what is called as a sensitive dependence on initial condition. And, in particular, (( )). He wanted to make it a little more dramatic, and he started talking about that so-called butterfly effect; the story goes according to this anecdote is that systems are so susceptible to small disturbances, that if a butterfly flaps its wings in Europe, it changes the weather in America. So, it is a kind of, a bit of a story, he wanted to sell it on a news magazine or coffee table thing, whether, it is Nature or Science does not matter, you have to shock the people; you have to give them some of this bylines, which sells.

So, this butterfly effect, and all those things came about. Now the question is, following those ideas, people found out that, if you decompose, let us say, Navier-Stokes equation, if you decompose Navier-Stokes equation using Galerkin projection, then, this truncated models, are shown to have this kind of properties. And, this is where you may have heard of, Lorentz. He was a meteorologist working in MIT, and he came out with this, and he basically produced some solution of Navier-Stokes equation, where it was truncated to four modes. And, he showed that, this four mode system was very susceptible to disturbances, and this chaos of, was interspersed by period of quiet. So, you have in the parameter space, you have a quiet flow followed by chaotic flow, then, again quiet flow, again chaotic flow, and this kind of things happened. And, there were other mathematicians, who came along and then, they said, look, you do not have to wait for this infinite sequence of instabilities as was proposed by Landau. They said, three sub-bifurcations are good enough, to take you from laminar to turbulent flow.

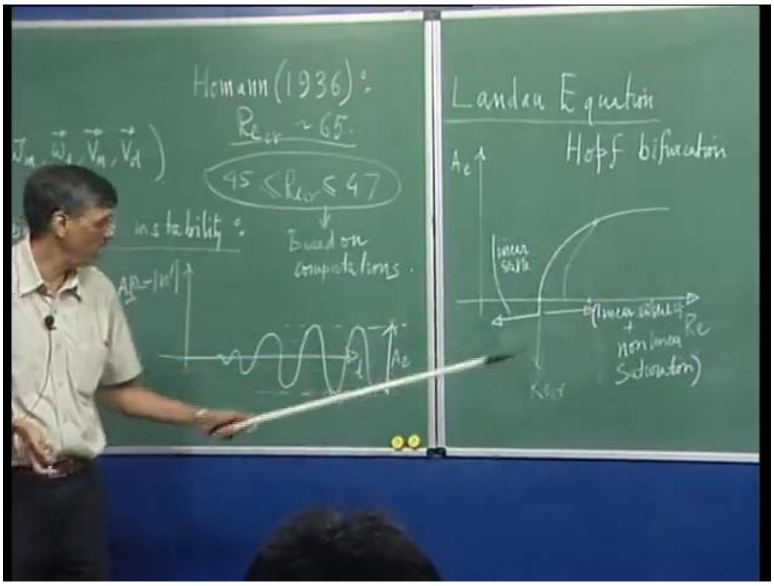
(Refer Slide Time: 42:46)



So, this was the kind of scenario with which we have grown up, in the last couple of decades. And it is very poignant, because people also tried to show this, with the help of fractal nature of turbulence, and this is the weak, you know, (( )), he was one of the pioneering mathematician who worked on fractals, he just passed away. So, it is a kind of a scenario, I am just trying to give you a big picture, where we stand today, that, there were, lots of this to and fros have gone in. But please, do understand that our approach to this discussion that we are talking about, is rooted to instability. We wanted to show, what instabilities are for unstable fluid dynamical systems, and as far as this phenomenon is concerned, or which is shown here, what is the role of non-linearity? Here, the role of non-linearity is not accentuating instability, it is moderating the instability. So, that is why, effect of non-linearity here is one of moderation, not of destabilization.

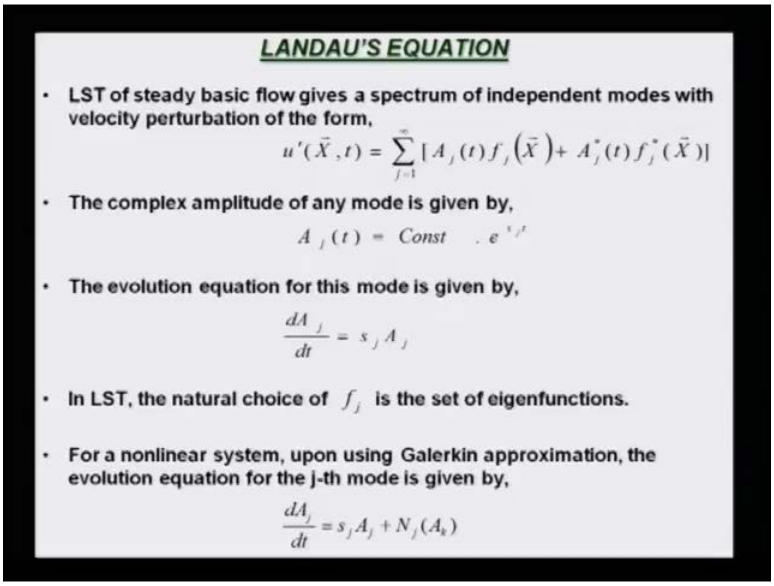
Now, Landau actually, interested, in trying to explain flows, which are known to be either completely stable, like Couette flow or pipe flow, we have talked about it, or let us say, in a channel flow, where the critical Reynolds number is known to be about 5772; but you do the experiment, as was done by Davis and White, in their proceedings, in Royal Society paper reported work; they found the flow in a channel becomes unstable as at a Reynolds number of as low as 1000. So, there is something that some of these flows suffer instability, while the linear theory says, they have to be stable.

(Refer Slide Time: 44:43)



So, whatever we see, that should be called as the sub-critical instability. So, Landau was developing this theory to explain sub-critical instability, but we explain only here, the super-critical stability; this is the case of super-critical stability. So, you are working on this side; so, flow has already become unstable, because you have crossed this Reynolds number; so, you are in the super-critical side and then, non-linearity actually saturates the amplitude. So, that is what we are talking about.

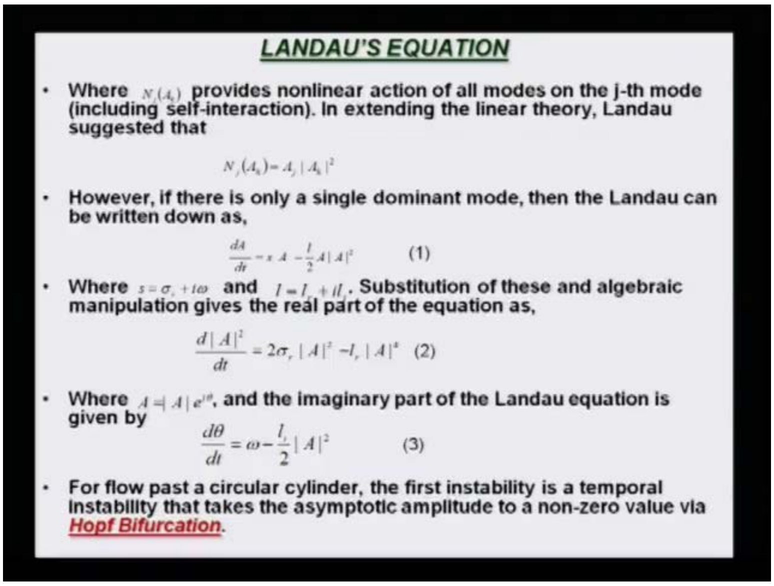
(Refer Slide Time: 45:07)



So, with this brief introduction, I think we can go ahead, and develop the Landau equation. What was done in Landau equation is that, if I am looking at the linear stability theory of steady basic flow, then, we know by now that we created large spectrum. So, the modes, let us say, they are independent and then, with the help of this modes, we can reconstruct the velocity perturbation in terms of this; this is your Galerkin projection, a time dependent function, time space dependent function, and we add the complex conjugate to make the whole thing itself as real. So, this A star, f star are nothing, but the complex conjugate of A and f. So, this is the usual way that we do. Now, the complex amplitude, when it has become unstable temporally, we could write that, A of t is some multiplicative constant times e to the power s j into t. So, what we are looking at, we are looking at the jth mode; and that has a growth or decay rate given by this complex exponent s j. Now, if this is the expression, then, I can differentiate it with respect to time; then, I will get d d t of A j is s j A j. This is what we will expect. Once I find those growth or decay rates as Eigen values, I can also, correspondingly find out the Eigen functions, and we can identify those f js with those Eigen functions.

So, starting from a linear theory, I could get the growth decay rate, I could get the Eigen functions and I get this. However, this is your linear description of the system. What happens is, we can obtain a corresponding non-linear system, and use the same kind of Galerkin approximation, and we would be able to show that the evolution equation, evolution equation means, differential equation with respect to time for the amplitude A j would be given like this. So, this is your linear part; this is the additional non-linear term, which we have symbolically written as N j. And, remember now, we have a large number of modes. I have written here infinite, but you have already seen, for Blasius flow, it just so happens here, we have only three such modes. So, it is not necessary that you will have to write it up to infinity; this is just for sake of completeness. Now what happens is, the jth mode, not only depends on the growth or decay rate of the jth mode from the linear stability analysis, but it can be, also affected by other modes. And, if you think of your Navier-Stokes equation, where would this term come from? This term would come from, let us say, something like your Reynolds stress like term, here. So, there you can see that different modes can interact, and that is the source of this. That is what we are saying, the non-linear effect on the jth mode, created by all possible kth mode. So, this is the general formalism or ((unsorts)) we can talk about.

(Refer Slide Time: 49:07)



Now, this is where Landau did his magic. Landau said that, this N j of A k does provide the non-linear action of all the modes on the jth mode, including self-interaction. The mode by itself can interact with itself. So, what Landau suggested that, this N j of A k is equal to A j times mod A k square. However, if we now talk about, only a single dominant mode, like what we have seen for all the cases, flow past a flat plate, we had a single mode which dictated the dynamics. Then, what happens is, this could be only coming from A j times A j mod square and then, I will have this kind of equation. And, there is this multiplicative constant l by two. Please note that, this s that we obtained from linear stability theory, we are looking at temporal instability; so, it will have a real part and imaginary part. Same way, this constant l, also will have a real and imaginary part. This l is what is called as the Landau coefficient. And, what we could do is, we could substitute this s and l expression here, and obtain the evolution equation for the amplitude square; this will be like this.

So, this is what was given in Landau's paper. He did not say, where he got this equation from; he just simply said the non-linearity is such, that it would be like this; because, he knew that, this equation is a close form solution. And, we will talk about it in the next class in greater detail, but whatever we are discussing here, this was very nicely elaborated by Stuart. And later on, this is given in that monograph by Drazin and Reid. You can read that; you can also read the lecture notes that I have written, where we have shown, how we get this equation, and we make this assumption that there is a single dominant mode.

And then, from there, we get, A, I could write it in terms of, in a polar form. Re to the power i theta, the R itself, I will call it as mod A. So, then, the governing equation for R will be like this, and governing equation for theta, will be like this. Landau actually completely missed this part, he never mentioned this. In fact, he wrongly said, you cannot even predict the phase. He has simply said, you can, at the most predict the amplitude and the equation would be given somewhat like this. So, I think this is where I would stop today, and we will begin from here tomorrow.