Introduction to Launch Vehicle Analysis and Design

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Lecture - 36
Optimal Multi-stage Solutions

Hello and welcome. In in this last tutorial that is tutorial number 4, we will look at the optimal staging configuration solution using the Lagrange multiplier method and we will understand the solution steps involved in the process. So let us begin.

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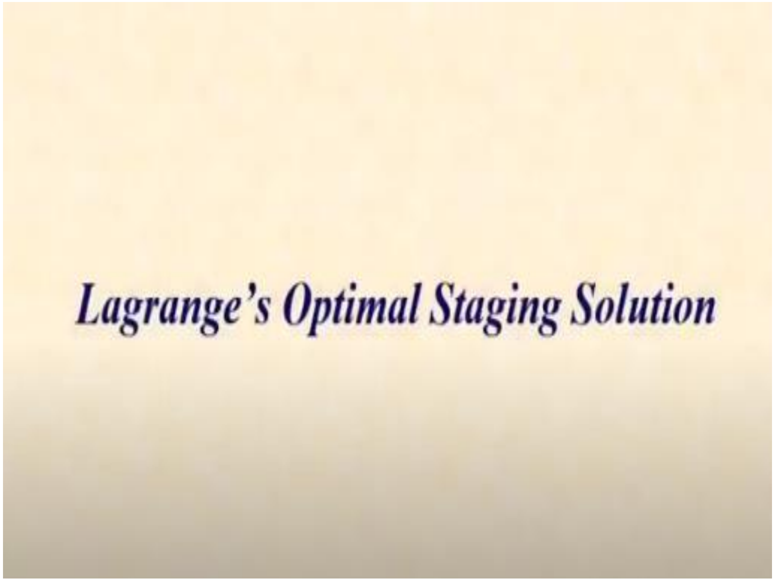

Let us begin our discussion with the simplified problem of equal stages.

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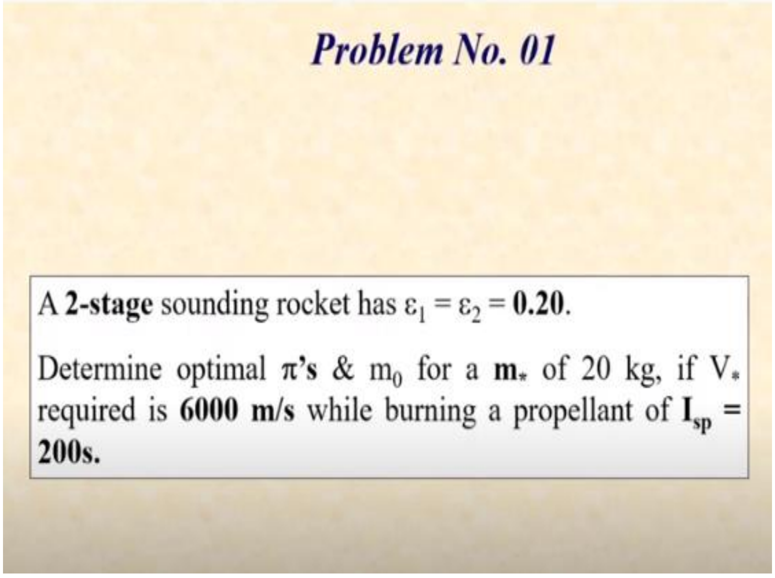

So, in this case let us consider a 2-stage sounding rocket, which has the structural ratio for both the stages as 0.2. Let us try and determine the optimal stage payload ratios that is $\pi's$ and the lift-off mass m_0 for an m_* or the payload of 20 kg if the ideal burnout velocity required in this case is 6000 m/s and it is burning a propellant of I_{sp} equal to 200 seconds.

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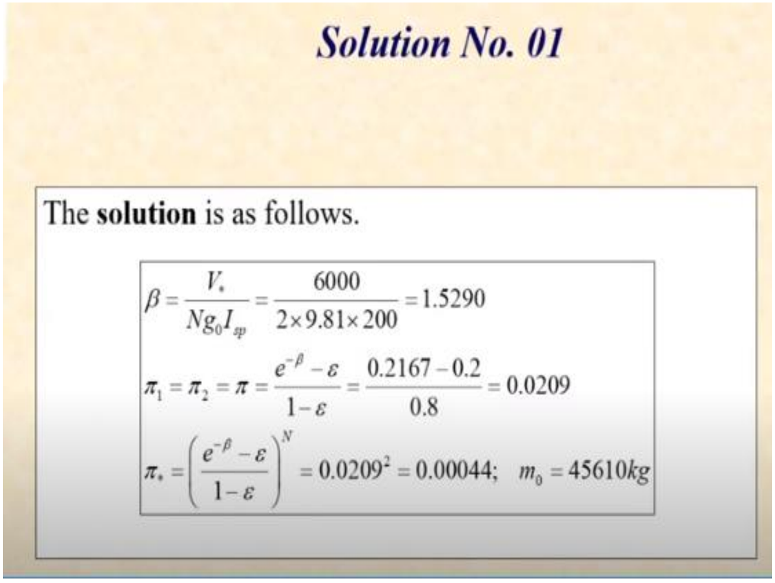

So let us start the solution. So, we first calculate the parameter β which is nothing but $\frac{V_*}{Ng_0 I_{sp}}$, where V_* is the desired velocity and are the number of stages and I_{sp} are the specific impulses. So, when we substitute these, we get β as 1.5290. Here you must keep in mind the fact that as we are going to use exponential functions, the number of significant digits have to be sufficient in order for us to capture the accuracy from such manipulations.

Otherwise, we may get an erroneous solution. So, in this case we have gone up to four decimal place and then we calculate the stage payload ratio that is π_1 and π_2 as $\frac{e^{-\beta}-\varepsilon}{1-\varepsilon}$. In the present case the ε is 0.2. And when we perform this calculation, we find that the 2-stage payload ratios are 0.0209, fairly small value.

Next, we talk about the mission payload fraction π_* which is nothing but a stage π^N where N is number of stages. So, in this case, as there are two stages, we get 0.0209^2 which is 0.00044. And from this, using the payload requirement of 20 kg we find that the lift-off mass is 45,610 kg or 45.6 tons. What it means is that our payload efficiency in this case is very small.

So just to launch a 20 kg payload, you are going to require a 45 tons rocket.

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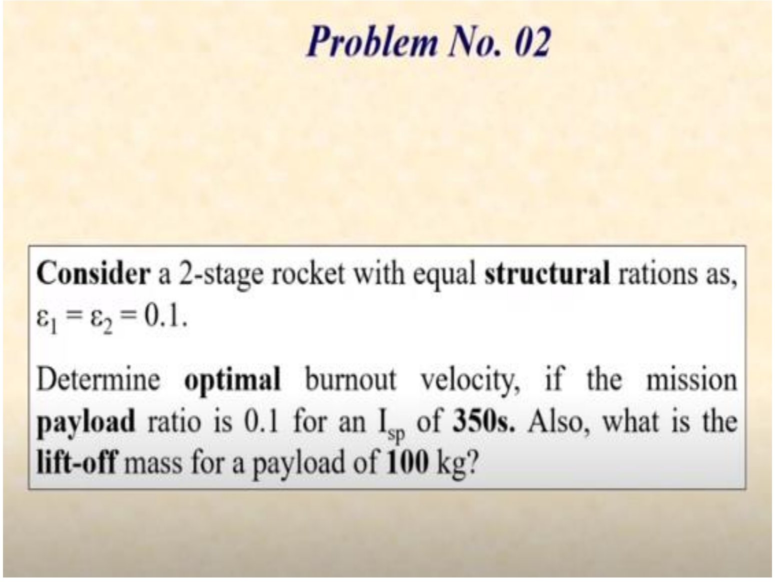

Let us now look at the problem of a 2-stage rocket again with the equal stages but this time we use the payload as the constraint saying that that payload is very small. Let us try and find out what is the burnout velocity that we can get and also, we have this time increase the I_{sp} from 200 to 350. So, we have improved the structure from 0.2 to 0.1.

So, it has become more efficient, it can carry more propellant and we have improved the I_{sp} from 200 to 350. And we also have a requirement of a significantly higher payload ratio of 0.1. And you also have a significantly higher requirement of payload of 100 kg. Now indirectly what we are saying is that we do not want the rocket lift-off mass to be more than 1 ton.

As against the 45 tons that we saw in the previous case, we would like our rocket mass to be only 1 ton. But we are willing to sacrifice the burnout velocity. Let us proceed with the solution.

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Solution No. 02

The **solution** is as follows.

$$\pi_1 = \pi_2 = \sqrt{\pi_*} = \sqrt{0.1} = 0.316$$

$$V_* = -g_0 I_{sp} N \ln\left[\varepsilon + (1-\varepsilon)\pi\right]$$

$$= -9.81 \times 350 \times 2 \times \ln\left[0.1 + 0.9 \times 0.316\right] = 6561.7 m/s$$

$$m_0 = \frac{100}{0.1} = 1000 kg$$

So, in this case the equal stage payload ratio from our formulae that we have seen in the lectures is $\sqrt{\pi_*}$ which is $\sqrt{0.1}$ and we find that the stage payload ratios have to be 0.316. Now with this stage payload ratio, we find that V_* or the ideal burnout velocity which is $-g_0 I_{sp} N \ln[\varepsilon + (1-\varepsilon)\pi]$.

If you substitute these numbers, we find that we get a velocity of 6,561.7 m/s which is a surprise. If we compare these two problems, we find that the previous problem a structural efficiency or ratio of 0.2 and an I_{sp} of 200 when it was changed to structural efficiency of 0.1 and the I_{sp} of 350 you are able to launch a 100 kg payload with 1 ton rocket and which also has a velocity which is higher than 6000 m/s.

Which obviously means that by small increases in the structural efficiency and small increases in propulsion technology, it is possible for us to significantly improve the efficiency of mission.

And you will find that the optimization techniques that we talk about essentially aim to look at slightly better structure, slightly better propulsion in order for us to have significantly better gains in terms of the launch cost which is typically expressed in terms of the payload fraction which is so many kgs of payload per kg of lift-off mass.

Let us now move over to a problem where we are going to have unequal stages. Which means the stages are not exactly equal.

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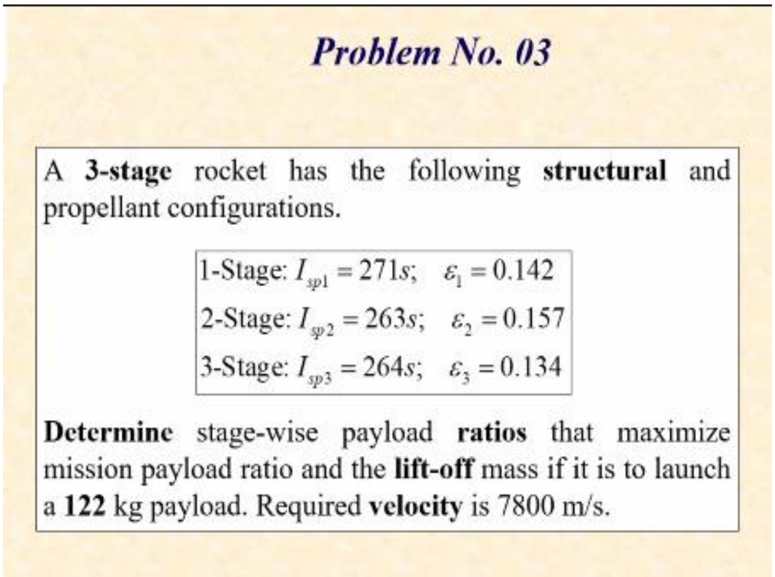

Problem No. 03

A **3-stage** rocket has the following **structural** and propellant configurations.

1-Stage: $I_{sp1} = 271s; \quad \varepsilon_1 = 0.142$

2-Stage: $I_{sp2} = 263s; \quad \varepsilon_2 = 0.157$

3-Stage: $I_{sp3} = 264s; \quad \varepsilon_3 = 0.134$

Determine stage-wise payload **ratios** that maximize mission payload ratio and the **lift-off** mass if it is to launch a **122** kg payload. Required **velocity** is 7800 m/s.

So, in this case, I have taken a 3-stage rocket which has marginally different I_{sp}'s and also marginally different structural efficiencies or structural ratios. You will find that these are typically the numbers that we would be getting in most practical rocket configurations. And our objective is to determine the stage-wise payload ratios, which will maximize the mission payload ratio such that a 122 kg payload is given a velocity of 7800 m/s.

In this context, it might be worth noting that the velocity which is specified in this problem corresponds to the spacecraft forming a circular orbit at 200 km altitude. So, this is how the spacecraft mission requirements would appear in the form of rocket configuration design. Let us proceed with the solution.

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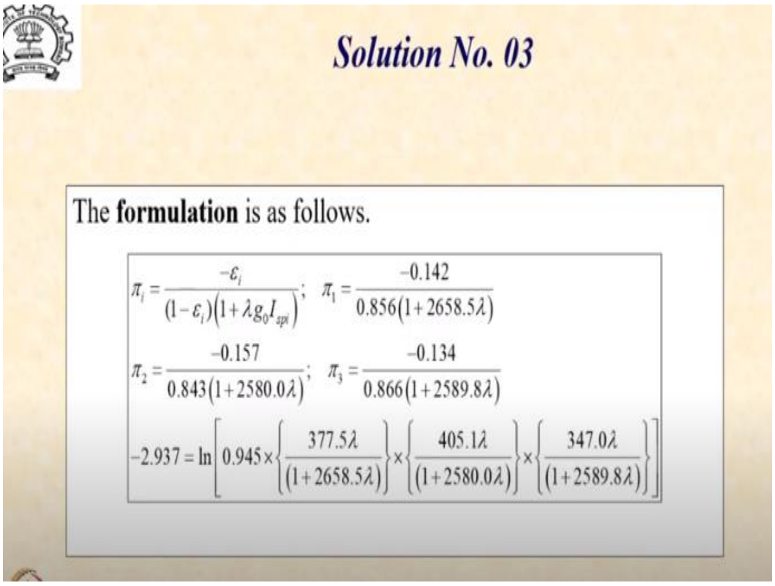

So first let us look at the basic formulation. So let us recall the solution for the stage payload ratio in the present case, which is $-\frac{\varepsilon_i}{(1-\varepsilon_i)(1+\lambda g_0 I_{spi})}$ where λ is the Lagrange multiplier and using the parameters that have been specified in the previous slide, let us now write down π_1, π_2 & π_3 in terms of λ.

So, π_1 is $-\frac{0.142}{0.856(1+2658.5\lambda)}$, which is nothing but this is the $g_0 I_{sp}$. Similarly, for π_2 we get $-\frac{0.157}{0.843(1+2580\lambda)}$. And π_3 is $-\frac{0.134}{0.866(1+2589.8\lambda)}$. Now the next step is to bring in the velocity constraint relation.

So here I suggest that you go and look up in the lecture the velocity constraint relation which is given as $V_* = -g_0 \sum I_{spi} \ln\left(\frac{\varepsilon_i \lambda g_0 I_{spi}}{1+\lambda g_0 I_{spi}}\right)$. This is the expression that you can verify from the lecture. What we do is we substitute the applicable parameters into that expression and we do one more step.

We divide the left-hand side that is the velocity with $-g_0$ and also divide by the I_{sp1} that is 271. So, if we do that, on the left-hand side we are going to get -2.937. And then on the right-hand side what we are going to get will be the ratios of the two is I_{sp}'s with the I_{sp1} and that resulting ratio of the two stages is 0.945. This you can independently verify.

And then of course, we have $\varepsilon_1 g_0 I_{sp1}, \varepsilon_2 g_0 I_{sp2}$ and $\varepsilon_3 g_0 I_{sp3}$ as the numerator. And $1 + g_0 I_{sp}\lambda$ for first stage. $1 + g_0 I_{sp2}\lambda$ and $1 + g_0 I_{sp3}\lambda$. So, this is now the constraint relation that we must first solve to generate the value of the Lagrange multiplier λ. Let us now look at how we are going to get this.

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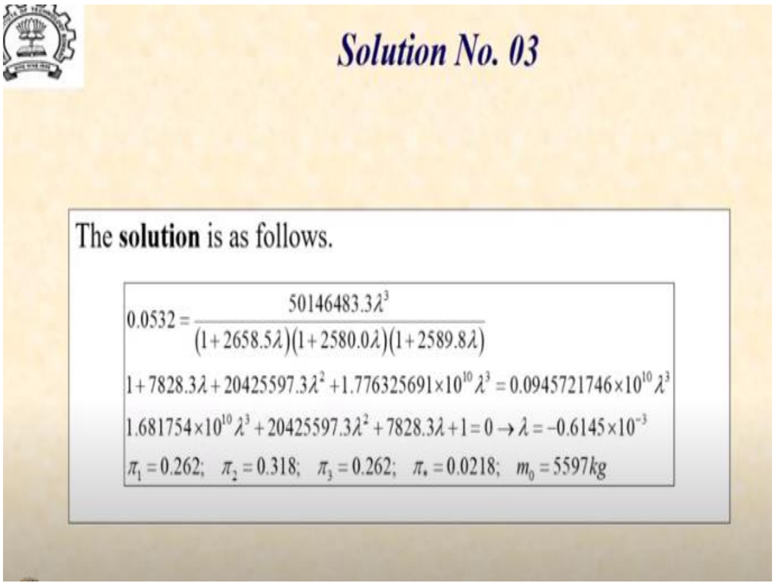

So, what we do is first of all, we remove the natural logarithm from right hand side by taking the exponential on the left-hand side that gives us 0.0532. And then of course, we perform all those multiplications so we start getting large numbers. Here again I must emphasize the need to retain all the significant digits, because as large numbers are involved, any truncation of number here is going to lead to large errors.

So, you see that when we expand the denominator, we are retaining the terms of the almost up to eighth or ninth decimal place, so, that the significant digits are all retained in the expansion. This results in the following cubic algebraic equation in λ that is $1.681754 \times 10^{10}\lambda^3 + 20425597.3\lambda^2 + 7828.3\lambda + 1 = 0$.

Now any standard numerical solver can be used including MATLAB to extract the roots. Please note that as is the cubic equation, there are going to be three roots of which only one of them will be the valid or the feasible solution root. So, in this case, we find that the one real root λ is -0.6145×10^{-3}. While the other two roots appear in the form of a complex conjugate and hence, they are invalid.

What we now do is take this value of λ and go back to these three expressions of $\pi_1, \pi_2\ and\ \pi_3$. Now you can clearly see that λ is a negative quantity. So, when you multiply this, you are going to get a minus number which is greater than 1, that can be confirmed. So, this denominator will become negative. This negative will cancel the numerator and we will get a positive number for $\pi_1, \pi_2\ and\ \pi_3$ as shown.

So, π_1 turns out to be 0.262, π_2 turns out to be 0.318 and π_3 turns out to be 0.262. We multiply all these three to get π_* which is 0.0218. And from this we can calculate the lift-off mass for 122 kg payload as about 5600 kg. Hi, so in this problem, we have seen that unequal stages problem where both epsilon and $I_{sp}'s$ are different in different stages, the numerical effort is primarily in setting up the equation for the Lagrange multiplier λ.

And you will find that the order of that equation will be equal to the number of stages and that setting up that problem and numerical solution will require some effort which is significantly larger than when we have equal stages. Now I will make a suggestion to all of you. Take problem number 3 the way we have defined and look at the structural ratio same for all the three stages as the geometric mean of the three values given in this example.

Which means we are given $\varepsilon_1, \varepsilon_2, \varepsilon_3$. You multiply all the three, take cube root that is the geometric mean and use this in all the three stages. Similarly, take the three I_{sp} values and take their arithmetic mean that is sum all of these and divide by 3. Please note these actions carefully.

And now assuming that these are to be used as constants for all the three stages generate the solution of the stages in terms of the $\pi's$ for all the three stages and check that value with respect to the geometric mean of the three π values that we have obtained in this problem. I am sure you will get some surprising results.

I hope you can go on to this journey and discover that how the simplified formulations and solution techniques that we have developed for optimal configuration can provide realistic and practical rocket configurations with not much computational effort. With that we come to the end of this tutorial. So, bye and thank you.