Introduction to Launch Vehicle Analysis and Design

Prof. Ashok Joshi
Department of Aerospace Engineering
Indian Institute of Technology – Bombay

Lecture – 12
Constant Velocity Solution

Hello and welcome. In the last lecture, we had started the discussion on the gravity turn trajectory and, in that context, we had looked at one specific case of constant pitch rate-based gravity turn trajectory solutions and we found that these were quite easily obtained as well as useful. In this lecture, we will look at the next concept of gravity turn trajectory where we achieve a constant velocity along the trajectory.

And still achieve a reasonable amount of turning of the trajectory through appropriate modulation of the thrust and the mass. So, let us begin.

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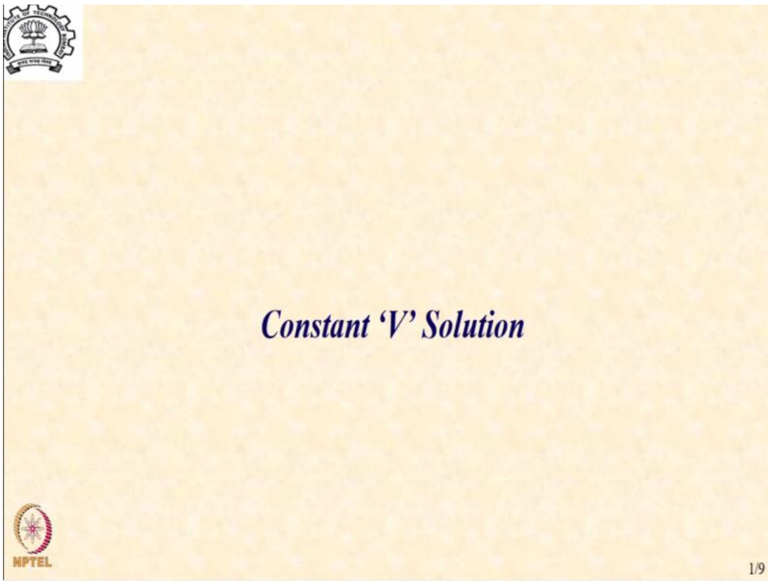

So, let us look at the various aspects of a constant velocity solution which will give us a reasonable amount of trajectory curvature and will also provide useful information.

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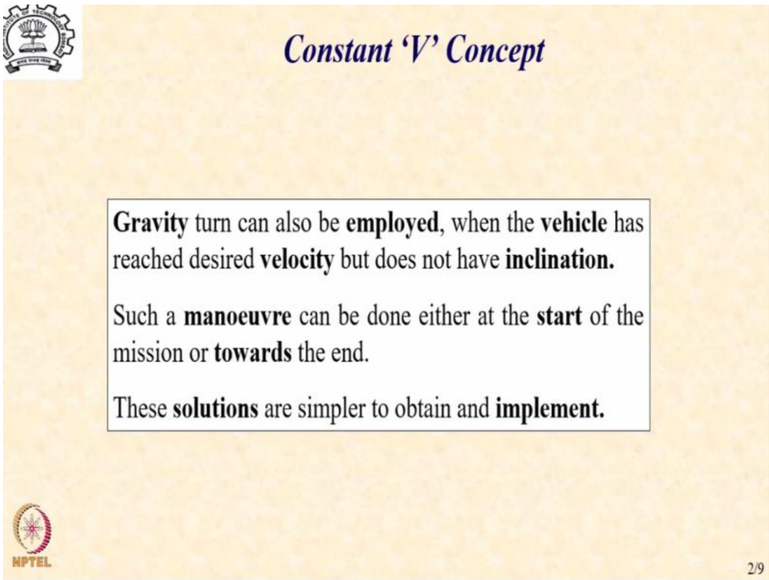

So, let us first understand why we would need such a scenario? While we see that gravity, turn is a mechanism by which we can achieve the trajectory curvature and inclination right after the lift off. There is a phase of the trajectory called the terminal phase where the vehicle is out of the atmosphere and also has achieved a very high velocity, but it is possible that it may not have the desired inclination or injection.

So, in such a case it is generally possible to employ a constant velocity gravity turn trajectory which helps us to achieve the desired inclination while maintaining the velocity at the desired level. Of course, as we will see these solutions are simpler to obtain as well as implement.

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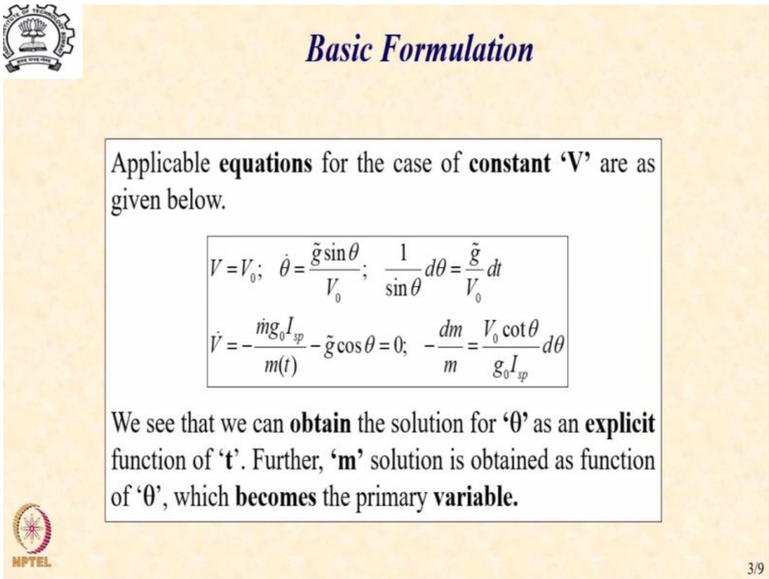

Let us now look at the basic formulation and the applicable equations for the constant V case. So, let us go back to our gravity turn equations and put the constant that $V = V_0$ a constant. We immediately see that our $\dot{\theta}$ equilibrium now has V_0 which is a constant value in the denominator. And we can now convert this directly into an equation between θ and t as $\frac{1}{\sin\theta} d\theta = \frac{\tilde{g}}{V_0} dt$.

We will see that integration of this equation will directly give θ as an explicit solution for time. So, this is one benefit that we get for constant velocity constraint. Further, when we look at the velocity equation the $\dot{V}$ becomes zero because it is a constant. And we now have the equation that $-\frac{\dot{m} g_0 I_{sp}}{m} - \tilde{g}\cos\theta = 0$. This equation now we can rewrite as an equation in terms of mass and θ.

So, now you will find that we can obtain mass as an explicit function of θ through this differential equation as $-\frac{dm}{m} = \frac{V_0 \cot\theta}{g_0 I_{sp}} d\theta$. So, you realize that the original two equations under the constraint that V is V_0 are modified into the two equations. One an explicit equation for θ in terms of time and the other equation for mass explicitly in terms of θ.

Now, we realize that in this context for the mass solution θ becomes the primary parameter and that is an important benefit particularly in the context where we have a requirement on inclination to be achieved at a given velocity by using the value of those inclinations, we can directly estimate the amount of mass or the propellant that will be required for that particular inclination to be achieved and that is a great design benefit.

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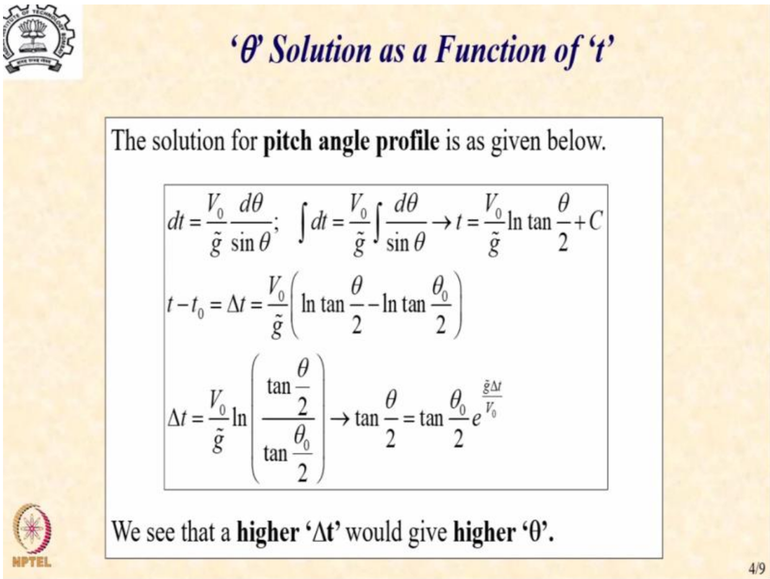

Now, of course we can solve these two differential equations. So, first let us solve θ as a function of t. So, it is a straight forward trigonometric integral equation and we can show that the time will be $\frac{V_0}{\tilde{g}} \ln \left(\frac{\tan\frac{\theta}{2}}{\tan\frac{\theta_0}{2}} \right)$ plus of course the constant of integration to be determined from initial condition which can be transformed into the Δt solution as given below.

Now this solution of course as we see is by directional. So, it can be worked in two ways. If you provide an inclination that is desired this particular equation will tell you how long it will take at that velocity to complete manoeuvre or the relation can be reversed and can be said that at that velocity if the manoeuvre is to be achieved in a certain time what is the angle that can be obtained.

And if the final angle is specified the same relation can also be used to determine the starting θ_0 from which this manoeuvre should be carried out and you will realize that this becomes an extremely useful design information. Of course, we also see that higher delta t would generally provide a higher θ, but again that will depend upon V_0. So, if you have a larger V_0 then we can appropriately modify this relation.

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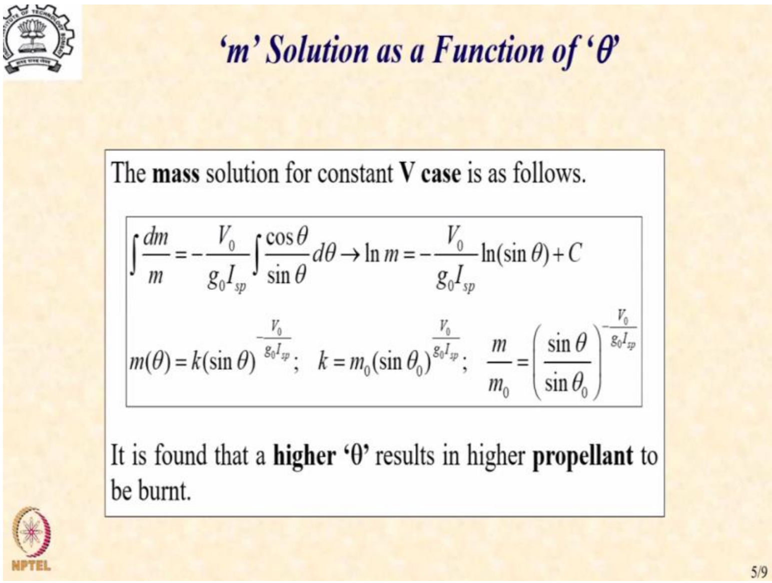

Next, let us now look at the solution of mass as a function of θ. So, we go back to the integral equation $\frac{dm}{m} = -\frac{V_0}{g_0 I_{sp}} \int \frac{\cos\theta}{\sin\theta} d\theta$ which is another simple trigonometric integral which can be obtained by a substitution of $\sin\theta$ as x and $\cos\theta \, d\theta$ as dx and because of which we can directly solve for this as $\ln m = -\frac{V_0}{g_0 I_{sp}} \ln \sin\theta + C$.

We can do little bit of algebraic manipulation on this expression and we can show that mass as a function of θ is the constant of integration $k(\sin\theta)^{-\frac{V_0}{g_0 I_{sp}}}$. When we substitute the initial condition that at $t = 0$ & $\theta = \theta_0$ we can obtain k as this expression so that we cannot talk about the mass fraction $\frac{m}{m_0} = \left(\frac{\sin\theta}{\sin\theta_0}\right)^{-\frac{V_0}{g_0 I_{sp}}}$.

Now, we clearly see that for a given V_0 and I_{sp} which means a given velocity and a given propellant if you want a higher correction in θ you will need to have a higher propellant because you will get a smaller $\frac{m}{m_0}$ indicating that there is a larger propellant mass that will be necessary to achieve that θ.

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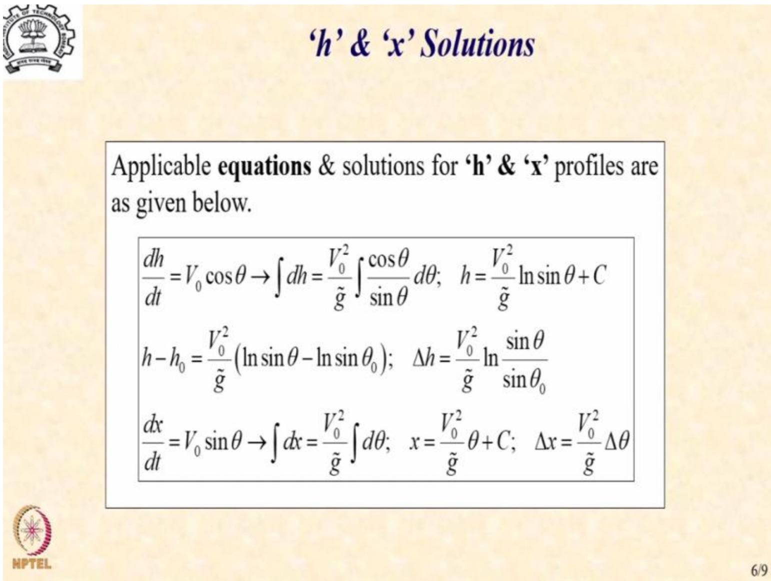

Let us now look at the trajectory solutions in terms of the h and x profiles. So, we go back to the kinematic equations that we have derived earlier that is $\frac{dh}{dt} = V_0 \cos\theta$ again it is an extremely simple trigonometric integral. So, we get a solution for h as $\frac{V_0^2}{\tilde{g}} \ln \sin\theta + c$. And the $\frac{dx}{dt} = V_0 \sin\theta$ is directly a function of $\frac{V_0^2}{\tilde{g}} d\theta$.

So, you realize that the expressions for h and x are quite simple. And of course, for higher $\Delta\theta$ you will travel a longer distance Δx and same thing will happen for a higher velocity.

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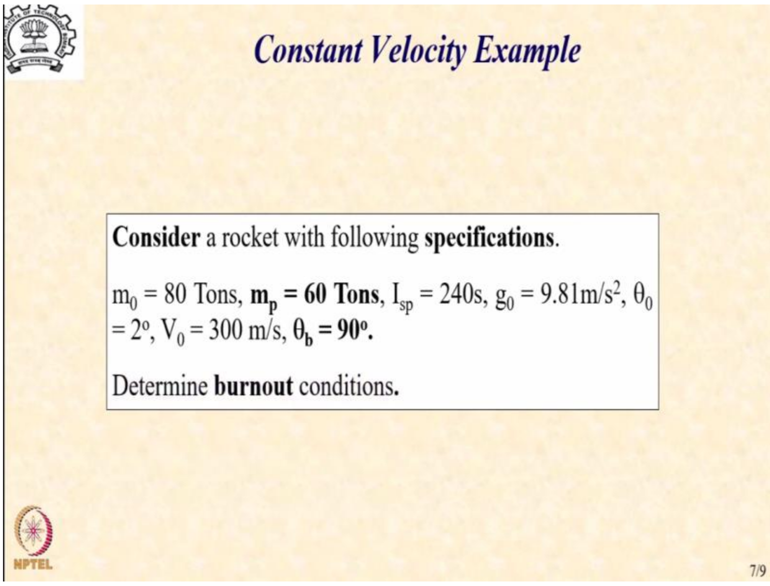

Let us now examine the implication of these relations for a problem that we have been considering. So, let us consider a rocket with following specifications. So, we have a lift off mass of 80 tons the propellant mass of 60 tons, for $I_{sp} = 240\text{s}$; $\theta_0 = 2°$ and $V_0 = 300\text{m}/s$ and the requirement is at we should become parallel to local horizon assuming flat earth approximation so $\theta_b = 90°$. Let us try and look at the solutions in terms of your mass profile, altitude profile and the horizontal distance profile.

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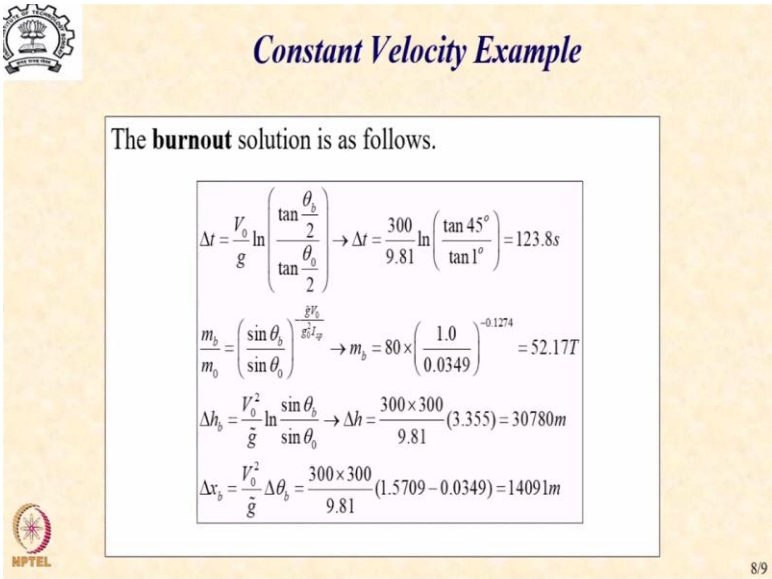

So, the time taken for this trajectory the first solution because we have given both θ_0 and θ_b at this V_0 it would take about 123.8s to complete the mission. And then we go to the mass profile that is $\frac{m}{m_0}$ or $\frac{m_b}{m_0}$ expression and it can be shown that the burnout mass in this case will be 52.17 tons. And if we start with 80 tons then effectively about 28 tons of propellant is going to be consumed during this.

So, you will consume about 28 tons of propellant in the time interval of about 123s to achieve an inclination from 2° to 90° at a constant velocity of 300 m/s. So, which means that you can actually achieve a significantly larger inclination by just burning a reasonably smaller amount of propellant as long as your velocity is constant. Here it is worth noting that this particular feature is due to the fact that you are not really using your propellant to accelerate the vehicle.

But only to turn the vehicle and this is directly responsible for the normal acceleration component which is coming because of gravity. So, you would realize that we can achieve large angular changes by burning a smaller amount of propellant and the results also justify our original hypothesis that constant velocity solutions manage the propellant better from a practical perspective.

Of course, because the trajectory is for a long duration of about 124s the altitude during this will of the order of around 31 km and it will travel about 14 km on the surface of the earth. As the distance travelled over the surface of earth is not very large during this period, a flat earth approximation with which we have started becomes reasonably applicable.

So, the results obtained in this case are fairly accurate and can be used for initial design and sizing exercise for such a mission.

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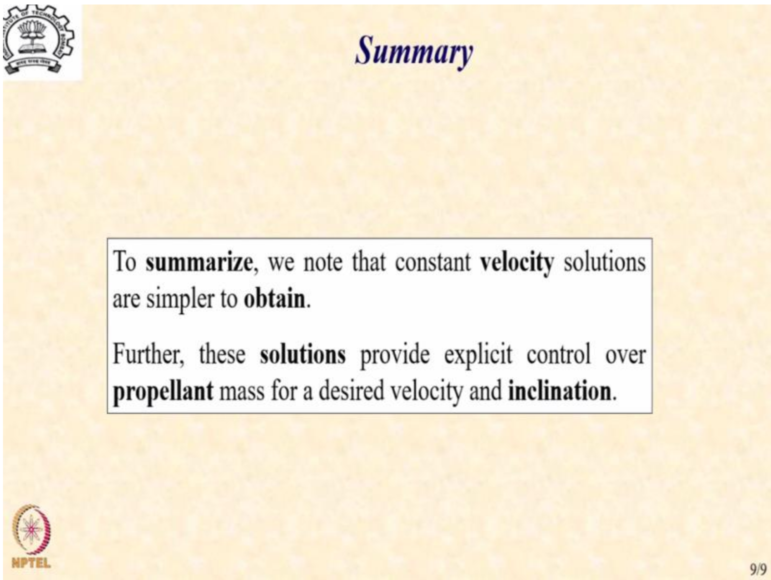

So, to summarize we note that a constant velocity solution is much simpler to obtain and obviously as the expressions are simpler the implementation also would become lot easier. And there is an important aspect which comes out very clearly is that there is now an explicit control over the propellant mass for a desired velocity and inclination which means if we specify a velocity and an inclination then it directly tells you how much of propellant you are going to need to complete the mission where at time would then come out as a natural consequence of the solution.

So, we have seen that the constant velocity solution for gravity turn is an extremely useful solution for minor trajectory corrections in the context of errors in inclination when the vehicle has reached its desired terminal velocity and the solution also indicates that the propellant that we are going to consume is kept to a minimum because we are not accelerating the vehicle.

With this, we will now look at in the next lecture the last of the simplified gravity turn solution trajectory which will be driven by a constant specific thrust or what is called constant forward acceleration which also has its own practical value. So, bye see you in the next lecture and thank you.